~~is determined by~~

$$\sum_{j=1}^{n} \widetilde{x}^j \widetilde{E}_j = \sum_{i=1}^{n} x^i E_i = \sum_{i,j=1}^{n} x^i A_i^j \widetilde{E}_j.$$

~~It follows that $\widetilde{x}^j = \sum_i A_i^j x^i$. Thus, the map sending x to $\widetilde{x}$ is an invertible linear map and hence a diffeomorphism, so any two such charts are smoothly compatible. The collection of all such charts thus defines a smooth structure, called the ***standard smooth structure on V***.~~ //

The Einstein Summation Convention

This is a good place to pause and introduce an important notational convention that is commonly used in the study of smooth manifolds. Because of the proliferation of summations such as $\sum_i x^i E_i$ in this subject, we often abbreviate such a sum by omitting the summation sign, as in

$$E(x) = x^i E_i, \quad \text{an abbreviation for } E(x) = \sum_{i=1}^{n} x^i E_i.$$

We interpret any such expression according to the following rule, called the ***Einstein summation convention***: if the same index name (such as i in the expression above) appears exactly twice in any monomial term, once as an upper index and once as a lower index, that term is understood to be summed over all possible values of that index, generally from 1 to the dimension of the space in question. This simple idea was introduced by Einstein to reduce the complexity of expressions arising in the study of smooth manifolds by eliminating the necessity of explicitly writing summation signs. We use the summation convention systematically throughout the book (except in the appendices, which many readers will look at before the rest of the book).

Another important aspect of the summation convention is the positions of the indices. We always write basis vectors (such as E_i) with lower indices, and components of a vector with respect to a basis (such as x^i) with upper indices. These index conventions help to ensure that, in summations that make mathematical sense, each index to be summed over typically appears twice in any given term, once as a lower index and once as an upper index. Any index that is implicitly summed over is a "dummy index," meaning that the value of such an expression is unchanged if a different name is substituted for each dummy index. For example, $x^i E_i$ and $x^j E_j$ mean exactly the same thing.

Since the coordinates of a point $\left(x^1, \dots, x^n\right) \in \mathbb{R}^n$ are also its components with respect to the standard basis, in order to be consistent with our convention of writing components of vectors with upper indices, we need to use upper indices for these coordinates, and we do so throughout this book. Although this may seem awkward at first, in combination with the summation convention it offers enormous advantages when we work with complicated indexed sums, not the least of which is that expressions that are not mathematically meaningful often betray themselves quickly by violating the index convention. (The main exceptions are expressions involving the Euclidean dot product $x \cdot y = \sum_i x^i y^i$, in which the same index appears twice in the upper position, and the standard symplectic form on $\mathbb{R}^{2n}$, which we will define in Chapter 22. We always explicitly write summation signs in such expressions.)

More Examples

Now we continue with our examples of smooth manifolds.

Example 1.25 (Spaces of Matrices). Let $M(m \times n, \mathbb{R})$ denote the set of $m \times n$ matrices with real entries. Because it is a real vector space of dimension mn under matrix addition and scalar multiplication, $M(m \times n, \mathbb{R})$ is a smooth mn-dimensional manifold. (In fact, it is often useful to *identify* $M(m \times n, \mathbb{R})$ with $\mathbb{R}^{mn}$, just by stringing all the matrix entries out in a single row.) Similarly, the space $M(m \times n, \mathbb{C})$ of $m \times n$ complex matrices is a vector space of dimension $2mn$ over $\mathbb{R}$, and thus a smooth manifold of dimension $2mn$. In the special case in which $m = n$ (square matrices), we abbreviate $M(n \times n, \mathbb{R})$ and $M(n \times n, \mathbb{C})$ by $M(n, \mathbb{R})$ and $M(n, \mathbb{C})$, respectively. //

Example 1.26 (Open Submanifolds). Let U be any open subset of $\mathbb{R}^n$. Then U is a topological n-manifold, and the single chart (U, Id_U) defines a smooth structure on U.

More generally, let M be a smooth n-manifold and let $U \subseteq M$ be any open subset. Define an atlas on U by

$$\mathcal{A}_U = \{\text{smooth charts } (V, \varphi) \text{ for } M \text{ such that } V \subseteq U\}.$$

Every point $p \in U$ is contained in the domain of some chart (W, φ) for M; if we set $V = W \cap U$, then $(V, \varphi|_V)$ is a chart in $\mathcal{A}_U$ whose domain contains p. Therefore, U is covered by the domains of charts in $\mathcal{A}_U$, and it is easy to verify that this is a smooth atlas for U. Thus any open subset of M is itself a smooth n-manifold in a natural way. Endowed with this smooth structure, we call any open subset an ***open submanifold of*** M. (We will define a more general class of submanifolds in Chapter 5.) //

Example 1.27 (The General Linear Group). The ***general linear group*** $\mathrm{GL}(n, \mathbb{R})$ is the set of invertible $n \times n$ matrices with real entries. It is a smooth n^2-dimensional manifold because it is an open subset of the n^2-dimensional vector space $M(n, \mathbb{R})$, namely the set where the (continuous) determinant function is nonzero. //

Example 1.28 (Matrices of Full Rank). The previous example has a natural generalization to rectangular matrices of full rank. Suppose $m < n$, and let $M_m(m \times n, \mathbb{R})$ denote the subset of $M(m \times n, \mathbb{R})$ consisting of matrices of rank m. If A is an arbitrary such matrix, the fact that rank $A = m$ means that A has some nonsingular $m \times m$ submatrix. By continuity of the determinant function, this same submatrix