Mémento maths : $\boxed{\text{Dérivées}}$

Notations : $\dfrac{dx}{dt} \equiv x'(t) \equiv \dot{x}(t)$

Dérivée composée :

$$\boxed{\frac{d}{dt}\left[f(x(t))\right] = \frac{df}{dx} \times \frac{dx}{dt}}$$

Exemple : $f(x) = x^2$

$$\Rightarrow \frac{d}{dt}\left[x^2(t)\right] = \frac{d(x^2)}{dx} \times \frac{dx}{dt} = 2x \times \dot{x}$$

Vérification sur un cas particulier :

$$x(t) = t^3$$

$$\frac{d}{dt}\left[x^2(t)\right] = 2x \cdot \dot{x} = 2t^3 \times 3t^2 = 6t^5$$

Or, $x^2(t) = t^6 \Rightarrow \dfrac{d}{dt}\left[x^2(t)\right] = \dfrac{d}{dt}(t^6) = 6t^5$

Applications : $\dfrac{d}{dt}(z^2) = \dfrac{d(z^2)}{dz}\dfrac{dz}{dt} = 2z\dot{z}$

$$\frac{d}{dt}(\dot{z}^2) = \frac{d(\dot{z}^2)}{d\dot{z}} \times \frac{d\dot{z}}{dt} = 2\dot{z}\ddot{z}$$

Dérivées impliquant des vecteurs :

$$\vec{r}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \boxed{\frac{d\vec{r}}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix}}$$

$$\boxed{\frac{d(\vec{a}\cdot\vec{b})}{dt} = \frac{d\vec{a}}{dt}\cdot\vec{b} + \vec{a}\cdot\frac{d\vec{b}}{dt}}$$

Preuve : $\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z$

$$\Rightarrow \frac{d}{dt}(\vec{a}\cdot\vec{b}) = \frac{d}{dt}(a_x b_x) + \frac{d}{dt}(a_y b_y) + \frac{d}{dt}(a_z b_z)$$

$$= \frac{da_x}{dt} b_x + a_x \frac{db_x}{dt} + \frac{da_y}{dt} b_y + a_y \frac{db_y}{dt}$$

$$+ \frac{da_z}{dt} a_z + a_z \frac{db_z}{dt}$$

$$= \frac{da_x}{dt} b_x + \frac{da_y}{dt} b_y + \frac{da_z}{dt} b_z$$

$$+ a_x \frac{db_x}{dt} + a_y \frac{db_y}{dt} + a_z \frac{db_z}{dt}$$

$$= \frac{d\vec{a}}{dt}\cdot\vec{b} + \vec{a}\cdot\frac{d\vec{b}}{dt}$$

$$\boxed{\frac{d(\vec{a}\wedge\vec{b})}{dt} = \frac{d\vec{a}}{dt}\wedge\vec{b} + \vec{a}\wedge\frac{d\vec{b}}{dt}}$$

Preuve : très similaire.

Application :

$$\frac{d\vec{a}^{\,2}}{dt} = \frac{d}{dt}(\vec{a}\cdot\vec{a}) = \frac{d\vec{a}}{dt}\cdot\vec{a} + \vec{a}\cdot\frac{d\vec{a}}{dt} = 2\vec{a}\cdot\frac{d\vec{a}}{dt}$$

Dérivées partielles :

Soit $V(x, y, z)$ une fonction de plusieurs variables.

$\boxed{\dfrac{\partial V}{\partial x}(x, y, z)}$ = dérivée de V par rapport à x en considérant y et z comme des paramètres.

Exemple : $V(x, y, z) = x^2 y + 3z$

$$\frac{\partial V}{\partial x} = 2xy \; ; \; \frac{\partial V}{\partial y} = x^2 \; ; \; \frac{\partial V}{\partial z} = 3$$

Dérivées partielles d'ordre 2 :

$$\boxed{\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial V}{\partial y}\right]}$$

Exemple : $\dfrac{\partial^2 V}{\partial x \partial y} = \dfrac{\partial}{\partial x}(x^2) = 2x$.

Théorème : $\boxed{\dfrac{\partial^2 V}{\partial x \partial y} = \dfrac{\partial^2 V}{\partial y \partial x}}$

Exemple :

$$\frac{\partial^2 V}{\partial y \partial x} = \frac{\partial}{\partial y}(2xy) = 2x.$$

Dérivées vectorielles :

$$\boxed{\vec{\nabla} V = \begin{pmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \\ \frac{\partial V}{\partial z} \end{pmatrix}}$$

Gradient

$$\boxed{\vec{\nabla} \wedge \vec{F} = \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{pmatrix}}$$

$$\vec{\nabla} \wedge \vec{F} = \overrightarrow{\text{rot}}\, \vec{F}$$

Rotationnel

Application : Si $\vec{F} = -\vec{\nabla} V$,

$\overrightarrow{\text{rot}}\, \vec{F} = \vec{0}$.

Preuve : conséquence immédiate de $\dfrac{\partial^2 V}{\partial x \partial y} = \dfrac{\partial^2 V}{\partial y \partial x}$.