



Differential Geometry II - Smooth Manifolds

Winter Term 2023/2024

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Exercise Sheet 3

Exercise 1 (*Equivalent characterizations of smoothness*):

Let M and N be smooth manifolds and let $F: M \rightarrow N$ be a map. Show that F is smooth if and only if either of the following conditions is satisfied:

- (a) For every $p \in M$ there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U \cap F^{-1}(V))$ to $\psi(V)$.
- (b) F is continuous and there exist smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N , respectively, such that for each α and β , $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

Exercise 2 (*Smoothness is a local property*):

Let M and N be smooth manifolds and let $F: M \rightarrow N$ be a map. Prove the following assertions:

- (a) If every point $p \in M$ has a neighborhood U such that $F|_U$ is smooth, then F is smooth.
- (b) If F is smooth, then its restriction to every open subset of M is smooth.

Exercise 3 (to be submitted by Friday, 13.10.2023, 20:00):

Let M , N and P be smooth manifolds. Prove the following assertions:

- (a) If $F: M \rightarrow N$ is a smooth map, then the coordinate representation of F with respect to every pair of smooth charts for M and N is smooth.
- (b) If $c: M \rightarrow N$ is a constant map, then c is smooth.
- (c) The identity map $\text{Id}_M: M \rightarrow M$ is smooth.
- (d) If $U \subseteq M$ is an open submanifold, then the inclusion map $\iota: U \hookrightarrow M$ is smooth.

- (e) If $F: M \rightarrow N$ and $G: N \rightarrow P$ are smooth maps, then the composite $G \circ F: M \rightarrow P$ is also smooth.

Exercise 4:

Let M_1, \dots, M_k be smooth manifolds. For each $i \in \{1, \dots, k\}$, let

$$\pi_i: \prod_{j=1}^k M_j \rightarrow M_i$$

be the projection onto the i -th factor.

- (a) Show that each π_i is smooth.
- (b) Let N be a smooth manifold. Show that a map $F: N \rightarrow \prod_{j=1}^k M_j$ is smooth if and only if each of the component maps $F_i := \pi_i \circ F: N \rightarrow M_i$ is smooth.

Exercise 5:

Let M be a smooth manifold of dimension $n \geq 1$. Show that the vector space $C^\infty(M)$ is infinite-dimensional.

[Hint: Show that if f_1, \dots, f_k are elements of $C^\infty(M)$ with non-empty disjoint supports, then they are linearly independent.]

Exercise 6:

Let A and B be disjoint closed subsets of a smooth manifold M . Show that there exists $f \in C^\infty(M)$ such that $0 \leq f(x) \leq 1$ for all $x \in M$, $f^{-1}(0) = A$ and $f^{-1}(1) = B$.