Prove the basic Theorem about Tens Décomp.

(Jennrich, Carroll, Harshman 1970's).

Basic Tool: Moore-Penrose pseudoinverse.

$A \in \mathbb{R}^{M \times N}$, $A^+ \in \mathbb{R}^{N \times M}$ real-matrix.

By def the MP pseudoinverse of A is the matrix $A^+ \in \mathbb{R}^{N \times M}$ satisfies.

1) $A A^+ A = A$

2) $A^+ A A^+ = A^+$

3) $(AA^+)^T = AA^+$ (so AA^+ sym matrix)

4) $(A^+A)^T = A^+A$ (idem).

Result of MP: A^+ always exists and is unique.

(of course in case A is invertible matrix $\to A^+ = A^{-1}$).

Properties of MP pseudo inverse (exercises).

1) A^+ exists and is unique.

2) If $A \in \mathbb{R}^{N\times M}$ then $A^+ \in \mathbb{R}^{M\times N}$.

3) $(A^+)^+ = A$

4) $(0_{N\times M})^+ = 0_{M\times N}$. ← zero matrix transposed.

↑ zero matrix

5) $(A^+)^T = (A^T)^+$

6) $(\alpha A)^+ = \alpha^{-1} A^+$, $\alpha \neq 0$.

↑ $\in \mathbb{R}$.

7) If A has full column rank & B has full row rank then $(AB)^+ = B^+A^+$.

8) A full column rank, then A^TA (which is sym) has full rank and is invertible (lin-algebra).
AND

$$A^+ = (A^TA)^{-1}A^T \quad \& \quad A^+A = I$$

9) A full row rank, then AA^T is full rank and invertible

AND $A^+ = A^T(AA^T)^{-1}$ & $AA^+ = I$.

Recall in thm $\quad R \leq \min(I_1, I_2)$

$A = [\underline{a}_1 \ldots \underline{a}_R] \quad I_1 \times R$

$B = [\underline{b}_1 \ldots \underline{b}_R] \quad I_2 \times R$ } full column rank.

$C = [\underline{c}_1 \ldots \underline{c}_R] \quad I_3 \times R$ ← pairwise indep of columns.

Thm states $T = \sum_{i=1}^{R} \underline{a}_i \otimes \underline{b}_i \otimes \underline{c}_i$ has a unique decomposition with at most R terms.
(up to trivial rescalings of $\underline{a}, \underline{b}, \underline{c}$).

Important idea of proof.

Given $T^{\alpha\beta\gamma}$ → form matrices by acting over vects.

$$[T(I, I, \underline{x})]^{\alpha\beta} = \underbrace{\sum_\gamma T^{\alpha\beta\gamma} x^\gamma}$$

$\underline{x} \in \mathbb{R}^{I_3}$

$$= \sum_{i=1}^{R} a_i^\alpha b_i^\beta \underbrace{\sum_\gamma c_i^\gamma x^\gamma}_{\underline{x}^T \cdot \underline{c}_i}$$

By looking at these Matrices along different vects $\underline{x}, \underline{y}, \ldots$
Reconstruct the tensor by Lin Alg Methods

Proof of Thm (Constructive proof → ACG)

- $\underline{x}$ & $\underline{y} \in \mathbb{R}^{I_3}$;

$$(T_{\underline{x}})^{\alpha\beta} = \sum_{i=1}^{R} a_i^{\alpha} b_i^{\beta} (\underline{x}^T \cdot \underline{c}_i)$$

$$T_{\underline{x}} = \sum_{i=1}^{R} \underline{a}_i \otimes \underline{b}_i \, (\underline{x}^T \cdot \underline{c}_i)$$

$$T_{\underline{y}} = \sum_{i=1}^{R} \underline{a}_i \otimes \underline{b}_i \, (\underline{y}^T \cdot \underline{c}_i) .$$

compute them from $T^{\alpha\beta\gamma}$

- Compute $T_x (T_{\underline{y}})^+$ and $(T_x)^+ T_y$

we will see eigenvectors of yields A & B .

$$T_x = \sum_{i=1}^{R} \underline{a}_i \underline{b}_i^T (\underline{x}^T \cdot \underline{c}_i) .$$

$$= \underbrace{[\underline{a}_1 \; \cdots \; \underline{a}_R]} \underbrace{\begin{bmatrix} \underline{x}^T \cdot \underline{c}_1 & & \\ & \ddots & \\ & & \underline{x}^T \cdot \underline{c}_R \end{bmatrix}}_{R \times R} \underbrace{\begin{bmatrix} \underline{b}_1^T \\ \underline{b}_2^T \\ \vdots \\ \underline{b}_R^T \end{bmatrix}}$$

$$= \underbrace{A}_{I_1 \times R} \; \underbrace{\mathrm{diag}(\underline{x}^T \cdot \underline{c}_r)}_{\substack{R \times R \\ r = 1 \cdots R}} \; \underbrace{B^T}_{R \times I_2} .$$

$I_1 \times I_2$
rank is R

// $\underline{x}$ s.t. no zero diag element.

$$(T_x)^+ = \left(A \operatorname{diag}(\underline{x}^T \cdot \underline{c}_r)\, B^T\right)^+$$

A full col rank
B^T full row rank.

$\underline{x}^T \cdot \underline{c}_r \neq 0$ with prob 1.

$\Big\}$ $\operatorname{diag}(\underline{x}^T \cdot \underline{c}_r)\, B^T$ full row rank.

<u>Prop 7</u>:

$$T_x^+ = \left(\operatorname{diag}(\underline{x}^T \cdot \underline{c}_r)\, B^T\right)^+ A^+$$
$$= (B^T)^+ \left(\operatorname{diag}\, \underline{x}^T \cdot \underline{c}_r\right)^+ A^+$$
$$= (B^T)^+ \operatorname{diag}\left(\frac{1}{\underline{x}^T \cdot \underline{c}_r}\right) A^+$$

similarly $\quad T_y^+ = (B^T)^+ \operatorname{diag}\left(\frac{1}{\underline{y}^T \cdot \underline{c}_r}\right) A^+$

(again $\underline{y}$ at random from $\mathbb{R}^{I_3}$ so w. prob 1 $\underline{y}^T \cdot \underline{c}_r \neq 0$)

• $T_x T_y^+ = A \operatorname{diag}(\underline{x}^T \cdot \underline{c}_r) \underbrace{B^T (B^T)^+}_{I \text{ since } B^T \text{ has full row rank.}} \operatorname{diag}\left(\frac{1}{y^T \cdot c_r}\right) A^+$

$$= A \operatorname{diag}\left(\frac{\underline{x}^T \cdot \underline{c}_r}{\underline{y}^T \cdot \underline{c}_r}\right) A^+$$

similarly

• $(T_x)^+ T_y = \underline{(B^T)^+} \operatorname{diag}\left(\frac{y^T \cdot c_r}{x^T \cdot c_r}\right) B^T$.

These two matrices have remarkable property:

$$\begin{cases} T_x T_y^+ A = A \operatorname{diag}\left(\frac{x^T \cdot c_r}{y^T \cdot c_r}\right) & \text{because } A^+ A = I \\ B^T T_x^+ T_y = \operatorname{diag}\left(\frac{y^T \cdot c_r}{x^T \cdot c_r}\right) B^T & B^T (B^T)^+ = I \end{cases}$$

Say that $\underline{a}_1, \underline{a}_2, \cdots, \underline{a}_R$ are e.vects of $T_x T_y^+$ with e.values $\frac{x^T \cdot c_r}{\underline{y}^T \cdot \underline{c}_r}$, $r = 1 \ldots R$. / Similar for $\underline{b}_1 \ldots \underline{b}_R$ e.vect of $T_x^+ T_y$ with

$$\begin{cases} (T_x T_y^{+})\, \underline{a}_r = \left(\dfrac{x^T \cdot c_r}{y^T \cdot c_r}\right) \underline{a}_r \\ \underline{b}_r^T\, T_x^{+} T_y = b_r^T \left(\dfrac{y^T \cdot c_r}{x^T \cdot c_r}\right). \end{cases}$$

$$\uparrow \quad (T_x^{+} T_y)^T\, \underline{b}_r = \left(\frac{y^T \cdot c_r}{x^T \cdot c_r}\right) b_r.$$

So you reconstruct $\underbrace{[\underline{a}_1 \ldots \underline{a}_R]}$ & $\underbrace{[\underline{b}_1 \ldots \underline{b}_R]}$ by solving two eigenvalue problems for $T_x T_y^{+}$ & $(T_x^{+} T_y)^T$

Since $\underline{c}_r$'s are pairwise independent easy to see that

$$\frac{x^T \cdot c_r}{y^T \cdot c_r} \neq \frac{x^T \cdot c_{r'}}{y^T \cdot c_{r'}} \qquad r \neq r'$$

(you would have $=$ if you had $\underline{c}_r = \lambda\, \underline{c}_{r'}$).

As a consequence eigen vects are unique (up to a scaling).

Remark that e.v for $\underline{a}_r$ is $\dfrac{\underline{x}^T \cdot c_r}{\underline{y}^T \cdot c_r}$

ev for $\underline{b}_r$ is $\dfrac{y^T \cdot c_r}{\underline{x}^T \cdot c_r}$ $\Big\} \to$ gives a way to pair $\underline{a}_r$ with $\underline{b}_r$.

- We have already constructed $A = [\underline{a}_1 \cdots \underline{a}_R]$

$$B = [\underline{b}_1 \cdots \underline{b}_R]$$

is unique way & we have correct pairing.

- Remains to be done is reconstruct $C = [\underline{c}_1 \cdots \underline{c}_R]$

$$\underbrace{T^{\alpha\beta\gamma}}_{I_1 I_2-\text{vector.}} = \sum_{r=1}^{R} \underbrace{a_r^\alpha \, b_r^\beta}_{} \, \underbrace{c_r^\gamma}_{R \text{ unknowns,}} \qquad (*)$$

Fix γ

$M_{\alpha\beta, r}$.

$\underbrace{I_1 I_2}_{\text{lines}} \times \underbrace{R}_{\text{column.}}$ matrix.

→ Linear syst of $I_1 I_2$ equations with R unknowns.

→ It is easy to check that $M_{\alpha\beta, r}$ $I_1 I_2 \times R$ is full column rank $= R$ (Exercise) because A & B have full column rank.

→ The syst of equations $(*)$ has a unique solution → c_r^γ

True for all γ separately.

Jenrich Algorithm.

(premise $T = \sum_{i=1}^{R} \underline{a}_i \otimes \underline{b}_i \otimes \underline{c}_i$).

Input $T^{\alpha\beta\gamma}$ $\alpha = 1 \ldots I_1$, $\beta = 1 \ldots I_2$, $\gamma = 1 \ldots I_3$

Output A, B, C

1) Compute $T_{\underline{x}}^{\alpha\beta} = \sum_\gamma T^{\alpha\beta\gamma} x^\gamma$ random $\underline{x} \in \mathbb{R}^{I_3}$

$T_y^{\alpha\beta} = \sum_\gamma T^{\alpha\beta\gamma} y^\gamma$ " $\underline{y} \in \mathbb{R}^{I_3}$

2) MP pseudoinverses $(T_x)^+$; $(T_y)^+$ & $T_x (T_y)^+$; $(T_x)^+ T_y$

3) Compute eigenvectors $\underline{a}_1 \ldots \underline{a}_R$ & $\underline{b}_1 \ldots \underline{b}_R$

pair them according to their inversely related eigenvalues

4) Solve for c_r^γ, $r = 1 \ldots R$, for each $\gamma = 1 \ldots I_3$

$$T^{\alpha\beta\gamma} = \sum_r \underbrace{a_r^\alpha b_r^\beta}_{\text{known by (3)}} c_r^\gamma$$

END / NEXT TIME present other algorithms.