Solutions to Homework 9

Exercise 1.

a) Fix $\epsilon > 0$. Then

$$\mathbb{P}(|X_n - C| > \epsilon) = \mathbb{P}(X_n < C - \epsilon) + \mathbb{P}(X_n > C + \epsilon) \leq F_{X_n}(C - \epsilon) + (1 - F_{X_n}(C + \epsilon)) \underset{n\to\infty}{\to} 0$$

whenever $X_n \overset{d}{\underset{n\to\infty}{\to}} C$.

b) Assume that $r > s \geq 1$ and that

$$\mathbb{E}(|X_n - X|^r) \underset{n\to\infty}{\to} 0.$$

Similar to exercise 4 a) on the midterm, we can use Jensen's inequality to show

$$\mathbb{E}(|X_n - X|^s)^{\frac{r}{s}} \leq \mathbb{E}(|X_n - X|^r)$$

by applying the convex function $f(x) = x^{r/s}$. Then

$$\mathbb{E}(|X_n - X|^s)^{\frac{r}{s}} \leq \mathbb{E}(|X_n - X|^r) \underset{n\to\infty}{\to} 0 \Rightarrow \mathbb{E}(|X_n - X|^s)^{\frac{r}{s}} \underset{n\to\infty}{\to} 0 \Rightarrow \mathbb{E}(|X_n - X|^s) \underset{n\to\infty}{\to} 0.$$

And therefore $X_n \overset{L^s}{\underset{n\to\infty}{\to}} X$.

c) We have that

$$|\mathbb{E}(X_n) - \mathbb{E}(X)| = |\mathbb{E}(X_n - X)| \leq \mathbb{E}(|X_n - X|) \underset{n\to\infty}{\to} 0.$$

Therefore $\mathbb{E}(X_n) \underset{n\to\infty}{\to} \mathbb{E}(X)$.

The converse is not true. Consider the sequence $(X_n, n \geq 1)$ of i.i.d. Bernoulli(p) random variables, with $0 < p < 1$. Then $\mathbb{E}(X_n) \underset{n\to\infty}{\to} \mathbb{E}(X)$. However

$$\mathbb{E}(|X_n - X|) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}.$$

This does not converge to zero as n goes to infinity.

Exercise 2. a) Let us compute first

$$\mathbb{E}(S_1) = \frac{1}{2}\left(\frac{3S_0}{2} + \frac{S_0}{2}\right) = S_0$$

Assuming now that $\mathbb{E}(S_n) = S_0$ (more precisely, that the expectation stays constant over n coin tosses), let us compute $\mathbb{E}(S_{n+1})$:

$$\begin{aligned}\mathbb{E}(S_{n+1}) &= \mathbb{E}(S_{n+1}|\{X_1 = +1\})\,\mathbb{P}(\{X_1 = +1\}) + \mathbb{E}(S_{n+1}|\{X_1 = -1\})\,\mathbb{P}(\{X_1 = -1\}) \\ &= \frac{1}{2}\left(\mathbb{E}(S_{n+1}|\{S_1 = \frac{3S_0}{2}\}) + \mathbb{E}(S_{n+1}|\{S_1 = \frac{S_0}{2}\})\right) = \frac{1}{2}\left(\frac{3S_0}{2} + \frac{S_0}{2}\right) = S_0\end{aligned}$$

Note: The computation is slightly unorthodox here, but we will see a cleaner way to prove this later in the course.

b) Y_n is the sum of n i.i.d. random variables, as the following computation shows:

$$Y_n = \log\left(\frac{S_n}{S_0}\right) = \log\left(\prod_{j=1}^n \left(1+\frac{X_j}{2}\right)\right) = \sum_{j=1}^n \log\left(1+\frac{X_j}{2}\right)$$

and these random variables are bounded, so by the central limit theorem,

$$\frac{Y_n - n\,\mu}{\sqrt{n}\,\sigma} \underset{n\to\infty}{\overset{d}{\to}} Z \sim \mathcal{N}(0,1)$$

where $\mu = \mathbb{E}(\log(1+X_1/2)) = \frac{1}{2}\,(\log(3/2)+\log(1/2)) \simeq -0.144$ and

$$\sigma^2 = \mathrm{Var}(\log(1+X_1/2)) = \frac{1}{2}\,(\log(3/2)^2+\log(1/2)^2) - \mu^2 \simeq 0.3$$

This is saying that for large n, we have

$$Y_n \simeq -0.144n + \sqrt{0.26n}\,Z \quad \text{in particular:} \quad Y_{100} \simeq -14.4 + 5.4\,Z$$

Therefore

$$\begin{aligned} \mathbb{P}(\{S_{100} > S_0/10\}) &= \mathbb{P}(\{S_{100}/S_0 > 1/10\}) = \mathbb{P}(\{Y_{100} > -\log(10)) \\ &\simeq \mathbb{P}\left(\left\{Z > \frac{-2.3+14.4}{5.4}\right\}\right) = \mathbb{P}(\{Z > 2.24\}) \end{aligned}$$

which is roughly 1% (so you can imagine what $\mathbb{P}(\{S_{100} > S_0\})$ looks like ...).

Therefore, the process $(S_n,\, n \geq 1)$, unexpectedly perhaps, "crashes" to zero with high probability as n gets large, even though it seemed a priori a "fair game" with constant expectation. This is an important example among a large class of processes called "martingales"; we will come back to this!

Note: The random process $(S_n,\, n \geq 1)$ is not unrelated to the following *deterministic* process defined recursively as

$$x_0 \in \mathbb{N}^*, \quad x_{n+1} = \begin{cases} x_n/2 & \text{if } x_n \text{ is even} \\ 3x_n+1 & \text{if } x_n \text{ is odd} \end{cases}$$

in which an even number gets multiplied by 1/2 and an odd number gets approximately multiplied by 3/2 (because it first gets multiplied by 3 and then necessarily divided by 2, as $3x_n+1$ is even). So if you consider that even and odd numbers appear naturally with probability 1/2, then the two processes have something in common. But in the deterministic case, one has no proof that the process ultimately reaches the value 1 as n gets large: this is the famous Collatz conjecture, which remains unsolved until now.

Exercise 3. a) let us compute $\mathbb{E}(S_n) = \sum_{j=1}^n \mathbb{E}(X_j^{(n)}) = n\,\frac{\lambda}{n} = \lambda$ and

$$\mathrm{Var}(S_n) = \sum_{j=1}^n \mathrm{Var}(X_j^{(n)}) = n\,\frac{\lambda}{n}\,\left(1-\frac{\lambda}{n}\right) = \lambda - \frac{\lambda^2}{n}$$

b) So $\mu = \lim_{n\to\infty} \mathbb{E}(S_n) = \lambda$ and $\sigma^2 = \lim_{n\to\infty} \mathrm{Var}(S_n) = \lambda$.

c) Let us compute the characteristic function of S_n:

$$\phi_{S_n}(t) = \mathbb{E}(\exp(itS_n)) = \mathbb{E}(\exp(it\,(X_1^{(n)} + \ldots + X_n^{(n)}))) = \mathbb{E}(\exp(itX_1^{(n)}))\cdots\mathbb{E}(\exp(itX_n^{(n)}))$$

$$= \left(\mathbb{E}(\exp(itX_1^{(n)}))\right)^n = \left(e^{it}\,\frac{\lambda}{n} + 1 - \frac{\lambda}{n}\right)^n = \left(1 + \frac{\lambda\,(e^{it}-1)}{n}\right)^n \underset{n\to\infty}{\to} \exp\left(\lambda\,(e^{it}-1)\right)$$

This limiting function is the characteristic function of $Z \sim \mathcal{P}(\lambda)$. Indeed, one can check that

$$\phi_Z(t) = \mathbb{E}(\exp(itZ)) = \sum_{k\geq 0} e^{itk}\,\frac{\lambda^k\,e^{-\lambda}}{k!} = e^{-\lambda}\sum_{k\geq 0}\frac{(\lambda\,e^{it})^k}{k!} = \exp(\lambda\,(e^{it}-1))$$

which allows us to conclude that $S_n \underset{n\to\infty}{\overset{d}{\to}} Z$.

d) The computation of the characteristic function is similar here:

$$\mathbb{E}\left(e^{itT_n}\right) = \left(\frac{1}{n}\,e^{it} + \left(1 - \frac{1}{n}\right)\right)^{\lceil \lambda n\rceil} = \left(1 + \frac{1}{n}\,(e^{it}-1)\right)^{\lceil \lambda n\rceil} \underset{n\to\infty}{\to} \exp(\lambda(e^{it}-1))$$

and leads actually exactly to the same result: T_n converges in distribution towards a Poisson random variable Z of parameter λ.

e) No, as each random variable S_n is constructed from a different set of random variables $X_1^{(n)}, \ldots, X_n^{(n)}$, which depends on n. The same holds for the random variables T_n.