Homework 9

Exercise 1. Let \((m_k, k \geq 0)\) be the sequence of moments of a generic random variable \(X\).

a) Let \(\ell \geq k \geq 0\). Show that if \(\mathbb{E}(|X|^\ell) < +\infty\) (i.e., if \(m_\ell\) is well-defined and finite), then it also holds that \(\mathbb{E}(|X|^k) < +\infty\).

Note: The reciprocal statement is that if \(\ell \geq k \geq 0\) and \(\mathbb{E}(|X|^k) = +\infty\), then \(\mathbb{E}(|X|^\ell) = +\infty\).

b) Show that the growth of the odd moments is controlled by the growth of the even moments.

c) Show that if \(X\) is bounded, then Carleman’s condition is satisfied:

\[
\sum_{k \geq 1} m_{2k}^{-\frac{1}{2k}} = +\infty
\]

Exercise 2. a) Let \(X \sim \mathcal{N}(0, 1)\). Compute all the moments of the random variable \(Y = \exp(X)\).

b) Let \(W\) be the discrete random variable such that

\[
\mathbb{P}(\{W = j\}) = C \exp(-j^2/2), \quad j \in \mathbb{Z}
\]

where \(C = 1/\sum_{j \in \mathbb{Z}} \exp(-j^2/2)\). Compute all the moments of the random variable \(Z = \exp(W)\).

c) What can you conclude from parts a) and b)?

Exercise 3*. Let \(X\) be a \(\mathcal{N}(0, \sigma^2)\) random variable, with \(\sigma > 0\).

a) Using integration by parts, show that for any continuously differentiable function \(f : \mathbb{R} \to \mathbb{R}\) such that there exists \(C > 0\) and \(n \geq 1\) with \(|f(x)|, |f'(x)| \leq C (1 + x^2)^n\) for every \(x \in \mathbb{R}\), we have

\[
\mathbb{E}(X \cdot f(X)) = \sigma^2 \mathbb{E}(f'(X))
\]

Note: The above condition is needed to ensure that both expectations are finite.

b) Use part a) to compute \(\mathbb{E}(X^k)\) for \(k \in \mathbb{N}\).

Let now \(m \geq 1\) and \(Y = X^m\).

c) For which values of \(m \geq 1\) and \(\sigma > 0\) does it hold that the distribution of \(Y\) is entirely determined by its moments? (Hint: use Stirling’s approximation: \(k! \simeq k^k e^{-k}\).
Exercise 4. Let \((X_n, n \geq 1)\) be a sequence of i.i.d. non-negative random variables defined on a common probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and such that there exists \(0 < a < b < +\infty\) with \(a < X_n(\omega) \leq b\) for all \(n \geq 1\) and \(\omega \in \Omega\). Let also \((Y_n, n \geq 1)\) be the sequence defined as

\[
Y_n = \left( \prod_{j=1}^{n} X_j \right)^{1/n}, \quad n \geq 1
\]

a) Show that there exists a constant \(\mu > 0\) such that \(Y_n \xrightarrow{n \to \infty} \mu\) almost surely.

b) Compute the value of \(\mu\) in the case where \(\mathbb{P}(\{X_1 = a\}) = \mathbb{P}(\{X_1 = b\}) = \frac{1}{2}\) and \(a, b > 0\).

c) In this case, look for the tightest possible upper bound on \(\mathbb{P}(\{Y_n > t\})\) for \(n \geq 1\) fixed and \(t > \mu\).

Hint. You have two options here. One is to use Chebyshev’s inequality with the function \(\psi(x) = x^p\) and \(p > 0\) (and then optimize over \(p\)) in order to upperbound

\[
\mathbb{P}(\{Y_n > t\}) = \mathbb{P}\left( \left\{ \prod_{j=1}^{n} X_j > t^n \right\} \right)
\]

for \(t > \mu\). The other option is left to your imagination...