Solutions to Homework 8

Exercise 1. a) For a given $\varepsilon > 0$, let us first consider n sufficiently large such that

$$\left|\frac{\mu_1 + \ldots + \mu_n}{n} - \mu\right| < \frac{\varepsilon}{2}$$

(such an n exists by assumption). For the same value of n, we have

$$\begin{aligned}
\mathbb{P}\left(\left\{\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right\}\right) &\leq \mathbb{P}\left(\left\{\left|\frac{S_n}{n} - \frac{\mu_1 + \ldots + \mu_n}{n}\right| \geq \frac{\varepsilon}{2}\right\}\right) \\
&= \mathbb{P}\left(\left\{\left|\sum_{i=1}^n (X_i - \mu_i)\right| \geq \frac{n\varepsilon}{2}\right\}\right) \leq \frac{4}{n^2\,\varepsilon^2}\,\mathbb{E}\left(\left(\sum_{i=1}^n (X_i - \mu_i)\right)^2\right) \\
&= \frac{4}{n^2\,\varepsilon^2}\sum_{i,j=1}^n \mathrm{Cov}(X_i, X_j) \leq \frac{4C_1}{n^2\,\varepsilon^2}\sum_{i,j=1}^n \exp(-C_2\,|i-j|) \\
&\leq \frac{8C_1}{n\,\varepsilon^2}\sum_{k\in\mathbb{Z}} \exp(-C_2\,|k|) \underset{n\to\infty}{\longrightarrow} 0, \quad \text{as} \quad \sum_{k\in\mathbb{Z}} \exp(-C_2\,|k|) < +\infty
\end{aligned}$$

b) We can check here that for $n \geq m \geq 1$, we have

$$\mathrm{Cov}(X_n, X_m) = a^{n-m}\,\mathrm{Var}(X_m)$$

and also that $\mathrm{Var}(X_1) = 0$ and

$$\mathrm{Var}(X_m) = 1 + a^2\,\mathrm{Var}(X_{m-1}) = \ldots = 1 + a^2 + a^4 + \ldots + a^{2(m-2)} \quad \text{for } m \geq 2$$

From this, we conclude that when $|a| < 1$, $\mathrm{Cov}(X_n, X_m)$ satisfies the condition given in the problem set. Besides, for every $n \geq 1$, we have

$$\mu_n = \mathbb{E}(X_n) = a\,\mathbb{E}(X_{n-1}) = a^{n-1}\,x$$

so

$$\lim_{n\to\infty} \frac{\mu_1 + \ldots + \mu_n}{n} = \frac{1}{n}\sum_{j=1}^n a^{j-1}\,x \underset{n\to\infty}{\to} 0$$

when $|a| < 1$, for any value of $x \in \mathbb{R}$. So $\mu = 0$ in this case and

$$\frac{S_n}{n} = \frac{X_1 + \ldots + X_n}{n} \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$$

Exercise 2. a) For $\varepsilon > 0$ and $n \geq 1$ fixed, let us compute, using the law of total probability:

$$\begin{aligned}
\mathbb{P}\left(\left\{\left|\frac{X_1 + \ldots + X_{T_n}}{T_n} - \mu\right| \geq \varepsilon\right\}\right) &= \sum_{k\geq 1} \mathbb{P}\left(\left\{\left|\frac{X_1 + \ldots + X_{T_n}}{T_n} - \mu\right| \geq \varepsilon\right\} \,\middle|\, \{T_n = k\}\right) \cdot \mathbb{P}(\{T_n = k\}) \\
&= \sum_{k\geq 1} \mathbb{P}\left(\left\{\left|\frac{X_1 + \ldots + X_k}{k} - \mu\right| \geq \varepsilon\right\} \,\middle|\, \{T_n = k\}\right) \cdot \mathbb{P}(\{T_n = k\}) \\
&= \sum_{k\geq 1} \mathbb{P}\left(\left\{\left|\frac{X_1 + \ldots + X_k}{k} - \mu\right| \geq \varepsilon\right\}\right) \cdot p_k^{(n)}
\end{aligned}$$

by independence of T_n and the sequence $(X_n,\, n \geq 1)$. From the proof of the weak law of large numbers, we know that for every $k \geq 1$:

$$\mathbb{P}\left(\left\{\left|\frac{X_1 + \ldots + X_k}{k} - \mu\right| \geq \varepsilon\right\}\right) \leq \frac{\sigma^2}{k\,\varepsilon^2}$$

so

$$\mathbb{P}\left(\left\{\left|\frac{X_1 + \ldots + X_{T_n}}{T_n} - \mu\right| \geq \varepsilon\right\}\right) \leq \frac{\sigma^2}{\varepsilon^2} \sum_{k \geq 1} \frac{p_k^{(n)}}{k}$$

A sufficient condition ensuring convergence in probability is therefore: $\lim_{n \to \infty} \sum_{k \geq 1} \frac{p_k^{(n)}}{k} = 0$.

b1) Let us compute for $n \geq 1$ and $k \geq 2$: (noting that the probability is equal to zero for $k = 1$)

$$\begin{aligned} p_k^{(n)} &= \mathbb{P}(\{T_n = k\}) = \sum_{j=1}^{k-1} \mathbb{P}(\{G_{n1} = j,\, T_n = k\}) = \sum_{j=1}^{k-1} \mathbb{P}(\{G_{n1} = j,\, G_{n2} = k - j\}) \\ &= \sum_{j=1}^{k-1} \mathbb{P}(\{G_{n1} = j\}) \cdot \mathbb{P}(\{G_{n2} = k - j\}) = \sum_{j=1}^{k-1} q_n^{j-1}\,(1 - q_n)\, q_n^{k-j-1}\,(1 - q_n) \\ &= (k-1)\, q_n^{k-2}\,(1 - q_n)^2 \end{aligned}$$

This implies that

$$\begin{aligned} \mathbb{E}(T_n) &= \sum_{k \geq 2} k\,(k-1)\, q_n^{k-2}\,(1 - q_n)^2 = \frac{\partial^2}{\partial z^2}\left(\sum_{k \geq 2} z^k\right)\Bigg|_{z = q_n} (1 - q_n)^2 \\ &= \frac{\partial^2}{\partial z^2}\left(\frac{1}{1-z} - 1 - z\right)\Bigg|_{z = q_n} (1 - q_n)^2 = \frac{2}{(1 - q_n)^3}\,(1 - q_n)^2 = \frac{2}{1 - q_n} \end{aligned}$$

Note: This result could also have been obtained using $\mathbb{E}(T_n) = \mathbb{E}(G_{n1}) + \mathbb{E}(G_{n2})$ together with the fact that a geometric random variable with parameter q has expectation $1/(1-q)$. [NB: geometric random variables with parameter q can be defined either on $\mathbb{N}^* = \{1, 2, 3, \ldots\}$ (as it is the case here) or on $\mathbb{N} = \{0, 1, 2, \ldots\}$, their expectation is equal to $q/(1-q)$ in the latter case]

b2) From the above computations, we see that

$$\sum_{k \geq 1} \frac{p_k^{(n)}}{k} = \sum_{k \geq 2} \frac{k-1}{k}\, q_n^{k-2}\,(1 - q_n)^2 \leq \sum_{k \geq 2} q_n^{k-2}\,(1 - q_n)^2 = \frac{1}{1 - q_n}\,(1 - q_n)^2 = 1 - q_n$$

so convergence in probability occurs if $q_n \underset{n \to \infty}{\to} 1$. This is in accordance with the fact that $\mathbb{E}(T_n) \underset{n \to \infty}{\to} +\infty$ in this case (see part a).

Exercise 3*. a) For all $x, y, z \in \mathbb{R}$ we have

$$\begin{aligned}
\log_2\left(1+\frac{|x-z|}{1+|x-z|}\right) &= \log_2\left(1+\frac{|x-y+y-z|}{1+|x-y+y-z|}\right) \\
&\leq \log_2\left(1+\frac{|x-y|+|y-z|}{1+|x-y|+|y-z|}\right) \\
&\leq \log_2\left(1+\frac{|x-y|}{1+|x-y|}+\frac{|y-z|}{1+|y-z|}\right) \\
&\leq \log_2\left(1+\frac{|x-y|}{1+|x-y|}\right)+\log_2\left(1+\frac{|y-z|}{1+|y-z|}\right)
\end{aligned}$$

where the first inequality follows from the fact that $\log_2(1+x)$ is an increasing function in x and the last inequality follows from the hint. Now, since the inequality holds for $X(\omega), Y(\omega), Z(\omega)$ for every $\omega \in \Omega$, we can take the expectation of both sides to get the desired result.

b) Fix $\epsilon > 0$ and note that convergence in probability implies that

$$\lim_{n\to\infty} \mathbb{P}(\{|X_n - X| \geq \epsilon\}) = 0.$$

For simplicity, define $g(x,y) = \log_2\left(1+\frac{|x-y|}{1+|x-y|}\right)$. We can write

$$\begin{aligned}
d(X_n, X) &= \mathbb{E}\left(g(X_n,X)1_{|X_n-X|\geq\epsilon|}\right) + \mathbb{E}\left(g(X_n,X)1_{|X_n-X|<\epsilon}\right) \\
&\leq \mathbb{E}\left(1_{|X_n-X|\geq\epsilon|}\right) + \log_2\left(1+\frac{\epsilon}{1+\epsilon}\right) \\
&= \mathbb{P}\left(\{|X_n-X|\geq\epsilon\}\right) + \log_2\left(1+\frac{\epsilon}{1+\epsilon}\right)
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} d(X_n, X) \leq \log_2\left(1+\frac{\epsilon}{1+\epsilon}\right).$$

Since this is true for any ϵ, we can further take a limit as ϵ goes to zero to get the desired result.

c) Yes, the converse is also true. Fix $\epsilon > 0$ and define $\nu = \log_2\left(1+\frac{\epsilon}{1+\epsilon}\right)$. Then

$$\begin{aligned}
\mathbb{P}\left(\{|X_n-X|\geq\epsilon\}\right) &= \nu \cdot \frac{1}{\nu}\mathbb{E}\left(1_{|X_n-X|\geq\epsilon|}\right) \\
&\leq \frac{1}{\nu}\mathbb{E}\left(g(X_n,X)1_{|X_n-X|\geq\epsilon}\right) \\
&\leq \frac{1}{\nu}d(X_n,X).
\end{aligned}$$

Since for a fixed ϵ, ν is just a constant, we have that

$$\lim_{n\to\infty} \mathbb{P}\left(\{|X_n-X|\geq\epsilon\}\right) = \frac{1}{\nu}\lim_{n\to\infty} d(X_n,X) = 0.$$