Homework 8

Exercise 1*. a) Let \((X_n, n \geq 1)\) be a sequence of bounded i.i.d. random variables such that \(E(X_1) = 0\) and \(\text{Var}(X_1) = 1\), and let \(S_n = X_1 + \ldots + X_n\) for \(n \geq 1\). Show that the event
\[
A = \left\{ \frac{S_n}{n} \text{ converges} \right\}
\]
belongs to the tail \(\sigma\)-field \(\mathcal{T} = \cap_{n \geq 1} \sigma(X_n, X_{n+1}, \ldots)\) (implying that \(P(A) \in \{0, 1\}\) by Kolmogorov’s 0-1 law; but the law of large numbers tells you more in this case, namely that \(P(A) = 1\)).

b) Assume now that \((X_n, n \geq 1)\) is a sequence of bounded, uncorrelated and identically distributed random variables such that \(E(X_1) = 0\) and \(\text{Var}(X_1) = 1\). The answer to Exercise 3.b) in homework 6 tells you what happens to \(P(A)\) in this case. That said, under this more general assumption, Kolmogorov’s 0-1 law may not necessarily hold. Prove it by exhibiting a sequence of random variables \((X_n, n \geq 1)\) satisfying these assumptions and an event \(B \in \mathcal{T}\) such that \(0 < P(B) < 1\).

Exercise 2. Someone proposes you to play the following game: start with an initial amount of \(S_0 > 0\) francs, of your choice. Then toss a coin: if it falls on heads, you win \(S_0/2\) francs; while if it falls on tails, you lose \(S_0/2\) francs. Call \(S_1\) your amount after this first coin toss. Then the game goes on, so that your amount after coin toss number \(n \geq 1\) is given by
\[
S_n = \begin{cases} 
S_{n-1} + \frac{S_{n-1}}{2} & \text{if coin number } n \text{ falls on heads} \\
S_{n-1} - \frac{S_{n-1}}{2} & \text{if coin number } n \text{ falls on tails}
\end{cases}
\]
We assume moreover that the coin tosses are independent and fair, i.e., with probability 1/2 to fall on each side. Nevertheless, you should not agree to play such a game: explain why!

Hints:
First, to ease the notation, define \(X_n = +1\) if coin \(n\) falls on heads and \(X_n = -1\) if coin \(n\) falls on tails. That way, the above recursive relation may be rewritten as \(S_n = S_{n-1} (1 + \frac{X_n}{2})\) for \(n \geq 1\).

a) Compute recursively \(E(S_n)\); if it were only for expectation, you could still consider playing such a game, but...

b) Define now \(Y_n = \log(S_n/S_0)\), and use the central limit theorem to approximate \(P(\{Y_n > t\})\) for a fixed value of \(t \in \mathbb{R}\) and a relatively large value of \(n\). Argue from there why it is definitely not a good idea to play such a game! (computing for example an approximate value of \(P(\{S_{100} > S_0/10\})\))

Exercise 3. Let \(\lambda > 0\) be fixed. For a given \(n \geq \lceil \lambda \rceil\), let \(X_1^{(n)}, \ldots, X_n^{(n)}\) be i.i.d. Bernoulli \((\lambda/n)\) random variables and let \(S_n = X_1^{(n)} + \ldots + X_n^{(n)}\).

a) Compute \(E(S_n)\) and \(\text{Var}(S_n)\) for a fixed value of \(n \geq \lceil \lambda \rceil\).

b) Deduce the value of \(\mu = \lim_{n \to \infty} E(S_n)\) and \(\sigma^2 = \lim_{n \to \infty} \text{Var}(S_n)\).
c) Compute the limiting distribution of $S_n$ (as $n \to \infty$).

*Hint:* Use characteristic functions. You might also have a look at tables of characteristic functions of some well known distributions in order to solve this exercise.

For a given $n \geq 1$, let now $Y_1^{(n)}, \ldots, Y_n^{(n)}$ be i.i.d. Bernoulli($1/n$) random variables and let

$$T_n = Y_1^{(n)} + \ldots + Y_{\lfloor \lambda n \rfloor}^{(n)}$$

where $\lambda > 0$ is the same as above.

d) Compute the limiting distribution of $T_n$ (as $n \to \infty$).

e) Is it also the case that either $S_n$ or $T_n$ converge almost surely or in probability towards a limit? Justify your answer!