**Solutions to Homework 12**

**Exercise 1.** a) We must check that $\mathbb{E}(\psi(Y)\, g(Y)) = \mathbb{E}(X\, g(Y))$ for any continuous and bounded function $g$. The computation gives indeed:

$$\mathbb{E}(\psi(Y)\, g(Y)) = \sum_{y\in C} \psi(y)\, g(y)\, \mathbb{P}(\{Y=y\}) = \sum_{x,y\in C} x\, g(y)\, \mathbb{P}(\{X=x, Y=y\}) = \mathbb{E}(X\, g(Y))$$

b) Let $Y$ and $Z$ be the two independent dice rolls: $\mathbb{P}(\{Y=i\}) = \mathbb{P}(\{Z=j\}) = 0.25$ and $\mathbb{P}(\{Y=i, Z=j\}) = \mathbb{P}(\{Y=i\})\,\mathbb{P}(\{Z=j\})$. We therefore have $\mathbb{E}(\max(Y,Z)|Y) = \psi(Y)$, where

$$\begin{aligned}\psi(i) &= \sum_{j=i}^{4} \max(i,j)\,\mathbb{P}(\{\max(Y,Z)=j\}|\{Y=i\}) = \sum_{j=i}^{4} \max(i,j)\,\frac{\mathbb{P}(\{\max(Y,Z)=j, Y=i\})}{\mathbb{P}(\{Y=i\})} \\ &= i\,\frac{\mathbb{P}(\{Z\le i, Y=i\})}{\mathbb{P}(\{Y=i\})} + \sum_{j=i+1}^{4} j\,\frac{\mathbb{P}(\{Z=j, Y=i\})}{\mathbb{P}(\{Y=i\})} = i\,\mathbb{P}(\{Z\le i\}) + \sum_{j=i+1}^{4} j\,\mathbb{P}(\{Z=j\})\end{aligned}$$

So $\psi(1)=2.5$, $\psi(2)=2.75$, $\psi(3)=3.25$ and $\psi(4)=4$.

**Exercise 2.**

a) From the previous Exercise,

$$\psi(n) = \sum_{k\ge 0} k\mathbb{P}(\{Z=k\}|\{N=n\}) = \sum_{k\ge 1}^{n} k\binom{n}{k} p^k (1-p)^{n-k} = pn$$

and so $\mathbb{E}(Z|N) = pN$. We also have that

$$\mathbb{E}(Z) = \mathbb{E}(\mathbb{E}(Z|N)) = \mathbb{E}(pN) = p\lambda.$$

b) We have that

$$\mathbb{P}(\{N=k\}|\{Z=z\}) = \frac{\mathbb{P}(\{N=k, Z=z\})}{\{Z=z\})} = \frac{\binom{k}{z} p^z q^{k-z} (\lambda^k/k!) e^{-\lambda}}{\sum_{m\ge z}\binom{m}{z} p^z q^{m-z}(\lambda^m/m!)e^{-\lambda}} = \frac{(q\lambda)^{k-z}}{(k-z)!} e^{-q\lambda}$$

From the previous Exercise,

$$\psi(z) = \sum_{k\ge 0} k\mathbb{P}(\{N=k\}|\{Z=z\}) = \sum_{k\ge z}^{n} k\frac{(q\lambda)^{k-z}}{(k-z)!} e^{-q\lambda} = z + q\lambda$$

and so $\mathbb{E}(N|Z) = Z + q\lambda$.

c) Intuitively, we would like each zombie to produce less than one offspring, so the infestation will die down if $p\lambda < 1$. However, even if this condition is met, we will also need $\lambda$ to be small (relative to the population of the planet). Otherwise, we risk having a few very hungry zombies eating the brains of some significant percentage of world's population before disappearing themselves.

**Exercise 3.** a) Let us compute theoretically the first three MSE's:

$$\mathbb{E}((\widehat{X}_1 - X)^2) = \mathbb{E}\left(\left(X + \frac{Z}{a} - X\right)^2\right) = \frac{\mathbb{E}(Z^2)}{a^2} = \frac{1}{a^2}$$

$$\mathbb{E}((\widehat{X}_2 - X)^2) = \mathbb{E}\left(\left(\frac{a^2 X + aZ}{a^2+1} - X\right)^2\right) = \mathbb{E}\left(\left(-\frac{1}{a^2+1}\, X + \frac{a}{a^2+1}\, Z\right)^2\right)$$

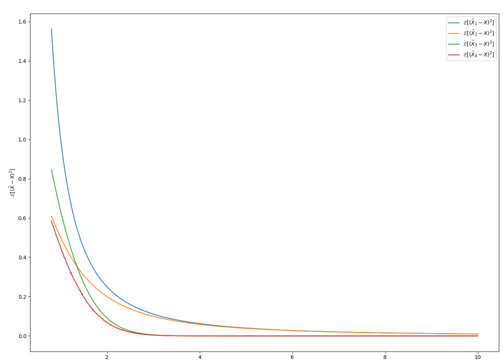
$$= \frac{1}{(a^2+1)^2} + \frac{a^2}{(a^2+1)^2} = \frac{1}{a^2+1}$$

$$\mathbb{E}((\widehat{X}_3 - X)^2) = \mathbb{E}((\text{sign}(a^2 X + aZ) - X)^2) = 4Q(-|a|)$$

where $Q(x) = \int_{-\infty}^{x} p_Z(z)dz$ is the cdf of $Z \sim \mathcal{N}(0,1)$. The fourth MSE is given by

$$\mathbb{E}((\widehat{X}_4 - X)^2) = \mathbb{E}((\tanh(a^2 X + aZ) - X)^2) = 1 - \mathbb{E}(\tanh(a^2 + aZ)^2)$$

(see computation below in part c) for the last equality). This expression can be computed by numerical integration or by the Monte-Carlo method, as suggested in the problem set. The four MSE's are represented as functions of $a$ on the figure below:




This shows that the fourth estimator gives the minimum MSE.

b) As we shall see below, the fourth estimator corresponds to the conditional expectation $\mathbb{E}(X|Y)$, which by definition minimizes $\mathbb{E}((Z-X)^2)$ among all random variables $Z$ which are $\sigma(Y)$-measurable and square-integrable. The computation of the condition expectation gives $\mathbb{E}(X|Y) = \psi(Y)$, where

$$\psi(y) = \sum_{x \in \{-1,+1\}} x\, \frac{p_x\, p_Z(y - ax)}{p_Y(y)} = \frac{p_Z(y-a) - p_Z(y+a)}{p_Z(y-a) + p_Z(y+a)} = \frac{e^{ay} - e^{-ay}}{e^{ay} + e^{-ay}} = \tanh(ay)$$

which confirms that $\mathbb{E}(X|Y) = \tanh(aY) = \widehat{X}_4$.

*NB:* The first expression for the function $\psi(y)$ above can be obtained either by reasoning intuitively (and forgetting that we are dealing here with a mix of discrete ($X$) and continuous ($Y$) random variables), or by proving formally that the random variable $\psi(Y)$ satisfies (similarly to Ex. 2.a):

$$\mathbb{E}(X g(Y)) = \mathbb{E}(\psi(Y)\, g(Y)), \quad \text{for every } g : \mathbb{R} \to \mathbb{R} \text{ continuous and bounded}$$

c) Using the following series of equalities:

$$\begin{aligned}\mathbb{E}((\mathbb{E}(X|Y) - X)^2) &= \mathbb{E}(X^2) + \mathbb{E}(\mathbb{E}(X|Y)^2) - 2\mathbb{E}(X\mathbb{E}(X|Y)) \\ &= \mathbb{E}(X^2) + \mathbb{E}(\mathbb{E}(X|Y)^2) - 2\mathbb{E}(\mathbb{E}(X\mathbb{E}(X|Y)|Y)) \\ &= \mathbb{E}(X^2) + \mathbb{E}(\mathbb{E}(X|Y)^2) - 2\mathbb{E}(\mathbb{E}(X|Y)^2) = \mathbb{E}(X^2) - \mathbb{E}(\mathbb{E}(X|Y)^2)\end{aligned}$$

we see that

$$\mathbb{E}((\widehat{X}_4 - X)^2) = \mathbb{E}(X^2) - \mathbb{E}(\widehat{X}_4^2) = 1 - \mathbb{E}(\tanh(aY)^2) = 1 - \mathbb{E}(\tanh(a^2 + aZ)^2)$$

(noticing for the last equality that the value of $X$ can be replaced by $+1$ using symmetry). A direct computation shows that it also holds that $\mathbb{E}((\widehat{X}_2 - X)^2) = \mathbb{E}(X^2) - \mathbb{E}(\widehat{X}_2^2) = \frac{1}{a^2+1}$, but that the equality does not hold for $\widehat{X}_1$ and $\widehat{X}_3$.

**Exercise 4.**

a) We know that $M_{n+1} - M_n \geq 0$ a.s., for all $n \geq 0$, and since $M$ is a martingale, we also know that $\mathbb{E}(M_{n+1} - M_n) = 0$ for all $n \geq 0$, so $M_{n+1} = M_n$ a.s. for all $n \geq 0$, i.e. $M_n = M_0$ a.s. for all $n \geq 0$.

b) Let us compute, for $n \geq 0$,

$$\begin{aligned}\mathbb{E}((M_{n+1} - M_n)^2) &= \mathbb{E}(M_{n+1}^2 - 2M_{n+1}M_n + M_n^2) = \mathbb{E}(\mathbb{E}(M_{n+1}^2 - 2M_{n+1}M_n + M_n^2|\mathcal{F}_n)) \\ &= \mathbb{E}(\mathbb{E}(M_{n+1}^2|\mathcal{F}_n) - 2\mathbb{E}(M_{n+1}|\mathcal{F}_n)M_n + M_n^2) = \mathbb{E}(M_n^2 - 2M_n^2 + M_n^2) = 0\end{aligned}$$

where we have used the assumption that $\mathbb{E}(M_{n+1}^2|\mathcal{F}_n) = M_n^2$. Therefore, $M_n = M_0$ a.s. for all $n \geq 0$.