Solutions to Homework 12

Exercise 1. a) By Jensen's inequality, we know that

$$\mathbb{E}(\varphi(M_{n+1})|\mathcal{F}_n) \geq \varphi(\mathbb{E}(M_{n+1}|\mathcal{F}_n))$$

We also know that $\mathbb{E}(M_{n+1}|\mathcal{F}_n) \geq X_n$, since $(M_n, n \in \mathbb{N})$ is a submartingale. The function φ needs therefore to be also increasing in order to ensure that

$$\mathbb{E}(\varphi(M_{n+1})|\mathcal{F}_n) \geq \varphi(M_n)$$

b) In particular, $(M_n^2,\, n \geq 1)$ need not be a submartingale, whereas $(\exp(M_n),\, n \geq 1)$ always is.

c) For every $n \in \mathbb{N}$, S_n^2 is bounded and $\mathcal{F}_n$-measurable, and

$$\begin{aligned}\mathbb{E}(S_{n+1}^2 \,|\, \mathcal{F}_n) &= \mathbb{E}(S_n^2 + 2S_n X_{n+1} + X_{n+1}^2 \,|\mathcal{F}_n) \\ &= S_n^2 + 2S_n \mathbb{E}(X_{n+1}) + \mathbb{E}(X_{n+1}^2) = S_n^2 + 1\end{aligned}$$

so we should take $c = 1$.

d) For every $n \in \mathbb{N}$, $\exp(S_n)/c^n$ is bounded and $\mathcal{F}_n$-measurable. Let us then compute

$$\begin{aligned}\mathbb{E}(\exp(S_{n+1}) \,|\, \mathcal{F}_n) &= \mathbb{E}(\exp(S_n + X_{n+1}) \,|\, \mathcal{F}_n) = \exp(S_n)\, \mathbb{E}(\exp(X_{n+1})) \\ &= \exp(S_n)\, \frac{e^{+1} + e^{-1}}{2} = \exp(S_n)\, \cosh(1)\end{aligned}$$

so we should take $c = \cosh(1)$.

e) The answer is no. Indeed, we have:

$$\mathbb{E}(S_{n+1}^2 \,|\, \mathcal{F}_n) = \mathbb{E}((S_n + X_{n+1})^2 \,|\, \mathcal{F}_n) = S_n^2 + 2S_n\, \mathbb{E}(X_{n+1}) + \mathbb{E}(X_{n+1}^2) = S_n^2 + 2(2p-1)\, S_n + 1$$

and there is no way to make the middle term disappear by considering $S_n^2 - cn$ with $c > 0$ instead of S_n^2.

f) In this case, the answer is yes:

$$\begin{aligned}\mathbb{E}(\exp(S_{n+1}) \,|\, \mathcal{F}_n) &= \mathbb{E}(\exp(S_n + X_{n+1}) \,|\, \mathcal{F}_n) = \exp(S_n)\, \mathbb{E}(\exp(X_{n+1})) \\ &= \exp(S_n)\, (p\, e^{+1} + (1-p)\, e^{-1})\end{aligned}$$

Taking therefore $c = p\, e^{+1} + (1-p)\, e^{-1}$, we obtain what we want.

Exercise 2*. a) To be added.

b) To be added

Exercise 3. Let $m \in \mathbb{N}$ and U be an $\mathcal{F}_m$-measurable and bounded random variable. Let us also define

$$H_n = \begin{cases} U, & \text{if } n = m+1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(H_n,\, n \in \mathbb{N})$ is predictable and for $m < N$, we have by assumption that M_m is $\mathcal{F}_m$-measurable and also that

$$0 = \mathbb{E}((H \cdot M)_N) = \mathbb{E}(U(M_{m+1} - M_m)).$$

Therefore, $M_m = \mathbb{E}(M_{m+1}|\mathcal{F}_m)$, so $(M_n,\, n \in \mathbb{N})$ is a martingale.

Exercise 4. For all $n \geq 0$, we have

$$\begin{aligned} &\mathbb{E}(M_{n+1}^2 - \mathbb{E}(M_{n+1}^2)|\mathcal{F}_n) = \mathbb{E}((M_n + X_{n+1})^2|\mathcal{F}_n) - \mathbb{E}((M_n + X_{n+1})^2) \\ &= M_n^2 - 2M_n\,\mathbb{E}(X_{n+1}) + \mathbb{E}(X_{n+1}^2) - \mathbb{E}(M_n^2) - 2\mathbb{E}(M_n X_{n+1}) - \mathbb{E}(X_{n+1}^2) = M_n^2 - \mathbb{E}(M_n^2) \end{aligned}$$

as $\mathbb{E}(M_n X_{n+1}) = \mathbb{E}(M_n)\,\mathbb{E}(X_{n+1}) = 0$.