Solutions to Homework 11

Exercise 1. We use the large deviations principle to find a tight upper bound. Before this, we need to check that the moment generating function $\mathbb{E}(e^{sX_1})$ is finite in a proper neighborhood of $s = 0$:

$$\mathbb{E}(e^{sX_1}) = \int_0^\infty e^{sx}\,\lambda e^{-\lambda x}\,dx = \frac{\lambda}{\lambda - s}, \quad \text{for } s < \lambda$$

Therefore, by applying the large deviations principle, we obtain for $t > 1/\lambda$:

$$\mathbb{P}(\{S_n > nt\}) \le \exp(-n\,\Lambda^*(t)) \quad \text{where} \quad \Lambda^*(t) = \max_{s\in\mathbb{R}} \left\{ st - \log\left(\frac{\lambda}{\lambda - s}\right)\right\}$$

By taking the derivative of $st - \log\left(\frac{\lambda}{\lambda-s}\right)$ with respect to s and setting it equal to zero, we obtain that $\Lambda^*(t)$ is maximum at $s^* = \lambda - \frac{1}{t}$. Hence,

$$\mathbb{P}(\{S_n > nt\}) \le \exp(-n\,(\lambda t - 1 - \log(\lambda t)))$$

Exercise 2.*

R **a)** For $X \sim \mathcal{N}(0, \sigma^2)$ we have

$$\begin{aligned} M_X(t) = \mathbb{E}(e^{tX}) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{tx} e^{-\frac{x^2}{2\sigma^2}}\,dx \\ &= \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\frac{t^2\sigma^2}{2}} \int_{-\infty}^{+\infty} e^{-\frac{(x-\sigma^2 t)^2}{2\sigma^2}}\,dx \\ &= \exp\left(\frac{t^2\sigma^2}{2}\right). \end{aligned}$$

b) For $X \sim \mathcal{U}([-a, a])$ we have

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_{-a}^{a} \frac{1}{2a} e^{tx}\,dx = \frac{1}{2at}(e^{ta} - e^{-ta}).$$

Now note that, using the Taylor expansion of e^x given in the hint, we can write

$$\begin{aligned} e^{ta} - e^{-ta} &= \sum_{n=0}^{\infty} \frac{(ta)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-ta)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(ta)^{2n+1}}{(2n+1)!} \\ &\le ta \sum_{n=0}^{\infty} \frac{(t^2a^2)^n}{2^n n!} \\ &= ta \exp\left(\frac{t^2a^2}{2}\right) \end{aligned}$$

where the inequality is due to the fact that $(2n+1)! \geq 2^n n!$, and the last equality is due to the Taylor expansion of $\exp\left(\frac{t^2a^2}{2}\right)$. Hence, we conclude that

$$M_X(t) \leq \frac{1}{2}\exp\left(\frac{t^2a^2}{2}\right) \leq \exp\left(\frac{t^2a^2}{2}\right).$$

c) By the Chebyshev-Markov inequality with $\psi(x) = e^{sx}$, we have

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(e^{sX})}{e^{st}} \leq \exp\left(\frac{s^2\eta^2}{2} - st\right).$$

The optimal s (which can be found by taking the derivative of the right-hand side and putting it equal to 0) is $s = \frac{t}{\eta^2}$, which we can substitute into the equation to get

$$\mathbb{P}(X \geq t) \leq \exp\left(\frac{t^2}{2\eta^2}\right).$$

The same upper-bound can be obtained similarly for $\mathbb{P}(X \leq -t)$, proving the result.

d) Note that, if Y_1 and Y_2 are two independent sub-gaussian random variables for some η_1 and η_2, then $Y_1 + Y_2$ is sub-gaussian with $\eta^2 = \eta_1^2 + \eta_2^2$. In fact,

$$M_{Y_1+Y_2}(t) = \mathbb{E}(e^{t(Y_1+Y_2)}) = \mathbb{E}(e^{tY_1})\mathbb{E}(e^{tY_2}) \leq \exp\left(\frac{t^2(\eta_1^2+\eta_2^2)}{2}\right).$$

One can apply this result recursively to prove the same property for the sum of n independent random variables. Then, the required result follows directly from part 3 with $X = \sum_{i=1}^n (X_i - \mathbb{E}(X_i))$.

Exercise 3. a) Use part (ii) of the definition with $U \equiv 1$ (such a U belongs to G).

b) (i) $Z = \mathbb{E}(X)$ is constant and therefore $\mathcal{G}$-measurable; (ii) Let $U \in G$: $\mathbb{E}(XU) = \mathbb{E}(X)\,\mathbb{E}(U) = \mathbb{E}(\mathbb{E}(X)\,U) = \mathbb{E}(ZU)$ (using the independence of X and U and the linearity of expectation).

c) (i) $Z = X$ is $\mathcal{G}$-measurable by assumption; (ii) Let $U \in G$: $\mathbb{E}(XU) = \mathbb{E}(ZU)$!

d) (i) $Z = \mathbb{E}(X|\mathcal{G})\,Y$ is $\mathcal{G}$-measurable; (ii) Let $U \in G$: $\mathbb{E}(XYU) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})\,YU)$, because part (ii) of the definition of $\mathbb{E}(X|\mathcal{G})$ implies the previous equality (indeed, $YU \in G$). Therefore, $\mathbb{E}(XYU) = \mathbb{E}(ZU)$.

e) Let us first check the left-hand side equality: $\mathbb{E}(X|\mathcal{H})$ is $\mathcal{H}$-measurable, therefore $\mathcal{G}$-measurable, so one can apply property c).

For the right-hand side equality, one has: (i) $Z = \mathbb{E}(X|\mathcal{H})$ is $\mathcal{H}$-measurable; (ii) Let $U \in H$:

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})\,U) = \mathbb{E}(\mathbb{E}(XU|\mathcal{G})) = \mathbb{E}(XU) = \mathbb{E}(\mathbb{E}(X|\mathcal{H})\,U) = \mathbb{E}(ZU)$$

using successively d), a) and the definition of $\mathbb{E}(X|\mathcal{H})$.