Homework 11

Exercise 1. Let $(X_n,\, n \geq 1)$ be a sequence of i.i.d. $\mathcal{E}(\lambda)$ random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., X_1 admits the following pdf:

$$p_{X_1}(x) = \begin{cases} \lambda \exp(-\lambda x) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Let also $S_n = X_1 + \ldots + X_n$. Using the large deviations principle, find a tight upper bound on

$$\mathbb{P}\left(\{S_n \geq nt\}\right) \quad \text{for } t > \mathbb{E}(X_1) = \frac{1}{\lambda}$$

Exercise 2.* Recall that the moment-generating function of a random variable X is defined for every $t \in \mathbb{R}$ as

$$M_X(t) = \mathbb{E}\left(e^{tX}\right).$$

a) Show that if $X \sim \mathcal{N}(0, \sigma^2)$, then

$$M_X(t) = \exp\left(\frac{1}{2}t^2\sigma^2\right).$$

We now introduce the concept of *sub-gaussianity.* A random variable X is called sub-gaussian if, for every $t > 0$,

$$M_X(t) \leq \exp\left(\frac{1}{2}t^2\eta^2\right)$$

for some $\eta \in \mathbb{R}^+$. (Note that η^2 need not be the variance of X!).

b) Show that if $X \sim \mathcal{U}([-a, a])$ for some $a > 0$, then X is sub-gaussian with $\eta = a$.

Hint: Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

c) Show that if X is sub-gaussian for some $\eta \in \mathbb{R}^+$, then for every $t > 0$,

$$\mathbb{P}(|X| \geq t) \leq 2\exp\left(-\frac{t^2}{2\eta^2}\right).$$

d) Prove the following generalization of Hoeffding's inequality. Let $X_i, i \in \{1, 2, \dots, n\}$ be independent random variables, where for each i, $X_i - \mathbb{E}(X_i)$ is sub-gaussian for some $\eta_i \in \mathbb{R}^+$. Let also $S_n = \sum_{i=1}^n X_i$. Show that for every $t > 0$,

$$\mathbb{P}(|S_n - \mathbb{E}(S_n)| \geq t) \leq 2 \exp\left(-\frac{t^2}{2\sum_{i=1}^n \eta_i^2}\right).$$

Exercise 3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X be an integrable random variable defined on this space and let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Relying only on the definition of conditional expectation, show the following properties:

a) $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$.

b) If X is independent of $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ a.s.

c) If X is $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$ a.s.

d) If Y is $\mathcal{G}$-measurable and bounded, then $\mathbb{E}(XY|\mathcal{G}) = \mathbb{E}(X|\mathcal{G})\, Y$ a.s.

e) If $\mathcal{H}$ is a sub-σ-field of $\mathcal{G}$, then $\mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G}) = \mathbb{E}(X|\mathcal{H}) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H})$ a.s.

Hint for parts b) to e): According to the course definition, in order to check that some candidate random variable Z is the conditional expectation of X given $\mathcal{G}$, you should check the following two conditions:

(i) Z is $\mathcal{G}$-measurable;

(ii) Z satisfies $\mathbb{E}((Z - X)\, U) = 0$ for every U $\mathcal{G}$-measurable and bounded.