Solutions to Homework 14

Exercise 1.*

a). First, lets show that if function $f : \mathcal{X}^n \to \mathbb{R}$ has c-bounded differences, it satisfies $1-$ Lipschitz condition.

i.e., Let $x, y \in \mathcal{Y}^n$ be such that they differ at the $j-$ th ($j \leq n$) position. Then, we have

$$|f(x) - f(y)| \leq c_j = d_c(x, y)$$

Now, lets extend it to $x, z \in \mathcal{Y}^n$ such that x and z differ at exactly two positions, say $i-$ th and $j-$ th index. Then, there exist a $y \in \mathcal{Y}^n$ such that it differs, from both x and z, only at 1 position. Then, we have

$$|f(x) - f(y)| \leq c_i$$
$$|f(y) - f(z)| \leq c_j$$

On adding both the equations and using triangle inequality, we have

$$|f(x) - f(z)| \leq |f(x) - f(y)| + |f(y) - f(z)| \leq c_i + c_j = d_c(x, z).$$

A further extension is possible for any two vectors which differ at at most n positions. Further, note that proving the converse i.e., $1-$ Lipschitz implies c-bounded differences, is not that hard..

b). Note that if $x \in \mathcal{Y}^n$, then the minimizer $y = x$. Thus, $d_c(x, x) = 0$ and consequently, $g(x) = f(x)$. For the case, when $x \notin \mathcal{Y}^n$, we have

$$\begin{aligned} g(x) &\leq \inf_{y \in \mathcal{Y}^n} f(y) + \sup_{y \in \mathcal{Y}^n} d_c(x, y) \\ &= \inf_{y \in \mathcal{Y}^n} f(y) + \sum_{i=1}^n c_i \\ &\leq \mathbb{E}(f(X)|X \in \mathcal{Y}^n) + \bar{c} \end{aligned}$$

Therefore,

$$g(X) \leq \begin{cases} f(X) & \text{if } X \in \mathcal{Y}^n \\ \mathbb{E}[f(X)|X \in \mathcal{Y}^n] + \bar{c} & \text{if } X \notin \mathcal{Y}^n \end{cases}$$

c). We know the following:

$$E(g(X)) = \mathbb{E}(g(X)|X \in \mathcal{Y}^n)\mathbb{P}(X \in \mathcal{Y}^n) + \mathbb{E}(g(X)|X \notin \mathcal{Y}^n)\mathbb{P}(X \notin \mathcal{Y}^n) \tag{1}$$

Using the result from part b)., we have:

$$\text{if } X \in \mathcal{Y}^n \; ; \; g(X) = f(X) \implies \mathbb{E}(g(X)|X \in \mathcal{Y}^n) = \mathbb{E}(f(X)|X \in \mathcal{Y}^n)$$
$$\text{if } X \notin \mathcal{Y}^n \; ; \; g(X) \leq \mathbb{E}(f(X)|X \in \mathcal{Y}^n) + \bar{c} \implies \mathbb{E}(g(X)|X \notin \mathcal{Y}^n) \leq \mathbb{E}(f(X)|X \in \mathcal{Y}^n) + \bar{c}.$$

Going back to Eq. (1), we have that:

$$\begin{aligned}E(g(X)) &\leq \mathbb{E}(f(X)|X \in \mathcal{Y}^n)\mathbb{P}(X \in \mathcal{Y}^n) + (\mathbb{E}(f(X)|X \in \mathcal{Y}^n) + \bar{c})\mathbb{P}(X \notin \mathcal{Y}^n) \\ &= \mathbb{E}((f(X)|X \in \mathcal{Y}^n) + p\bar{c}\end{aligned}$$

d). Let's begin with

$$\begin{aligned}\mathbb{P}[\big(f(X) - \mathbb{E}[f(X)|X \in \mathcal{Y}^n]\big) \geq \epsilon + p\bar{c}] &\leq \mathbb{P}(f(X) - \mathbb{E}(g(X)) \geq \epsilon) \\ &\leq \mathbb{P}(f(X) - \mathbb{E}(g(X)) \geq \epsilon | X \in \mathcal{Y}^n)\mathbb{P}(X \in \mathcal{Y}^n) + \mathbb{P}(X \notin \mathcal{Y}^n) \\ &\leq \mathbb{P}(g(X) - \mathbb{E}(g(X)) \geq \epsilon) + p \\ &\leq \exp\left(-\frac{\epsilon^2}{\sum_{i=1}^n c_i^2}\right) + p\end{aligned}$$

Hence, proved.

Exercise 2.
a) Observe first that for every value of $0 < p < 1$, we have $0 < M_{n+1} < 1$ whenever $0 < M_n < 1$ (and $M_0 = x \in]0,1[$ by assumption), so that it makes sense to consider both M_n and $1 - M_n$ as probabilities at every step. Let us then compute:

$$\mathbb{E}(M_{n+1} \,|\, \mathcal{F}_n) = pM_n\,(1 - M_n) + ((1-p) + pM_n)\,M_n = M_n$$

for every $0 < p < 1$!

b) Again, let us compute:

$$\begin{aligned}&\mathbb{E}(M_{n+1}\,(1 - M_{n+1})\,|\,\mathcal{F}_n) = pM_n\,(1 - pM_n)\,(1 - M_n) + ((1-p) + pM_n)\,(1 - (1-p) - pM_n)\,M_n \\ &= pM_n\,(1 - pM_n)\,(1 - M_n) + ((1-p) + pM_n)\,p(1 - M_n)\,M_n = p(2-p)\,M_n\,(1 - M_n)\end{aligned}$$

c) Therefore, we obtain by induction:

$$\mathbb{E}(M_n\,(1 - M_n)) = (p(2-p))^n\,x(1-x)$$

which converges to 0 as n gets large (as $p(2-p) < 1$ for every $0 < p < 1$).

d) Because the martingale M is bounded, the answer is yes to all three questions.

e) The answer obtained in c) suggests that M_n converges either to 0 or 1 as $n \to \infty$, which turns out to be the case. Moreover, as seen in class:

$$\mathbb{P}(\{M_\infty = 1\}) = \mathbb{E}(M_\infty) = \mathbb{E}(M_0) = x, \quad \text{so} \quad \mathbb{P}(\{M_\infty = 0\}) = 1 - x$$

Exercise 3.

a) As $0 < p < 1/2$, S is a supermartingale. Also, $\lambda^x = e^{x\log(\lambda)}$ is a convex function $\forall \lambda > 0$, but is increasing for $\lambda \geq 1$ and decreasing for $\lambda \leq 1$. Therefore, applying Jensen's inequality gives

$$\mathbb{E}(\lambda^{S_{n+1}} \,|\, \mathcal{F}_n) \geq \lambda^{\mathbb{E}(S_{n+1}\,|\,\mathcal{F}_n)} \geq \lambda^{S_n}$$

only when $\lambda \leq 1$.

b) We have

$$\mathbb{E}(\lambda^{S_{n+1}} \,|\, \mathcal{F}_n) = \lambda^{S_n}\, \mathbb{E}(\lambda^{X_{n+1}}) = \lambda^{S_n}\, (\lambda p + \frac{1}{\lambda}(1-p))$$

which says that Y is a submartingale if and only if $E = \lambda p + \frac{1}{\lambda}(1-p) \geq 1$. Solving this (quadratic) inequation, we obtain the condition: $\lambda \in]-\infty, 1] \cup [\frac{1-p}{p}, +\infty[$. For the martingale condition to hold ($E = 1$), it must be that either $\lambda = 1$ (trivial case) or $\lambda = \frac{1-p}{p}$. Finally, Y is a supermartingale ($E \leq 1$) if and only if $\lambda \in [1, \frac{1-p}{p}]$.

c) By independence, we have $\mathbb{E}(|Y_n|) = \mathbb{E}(Y_n) = \prod_{j=1}^n \mathbb{E}(\lambda^{X_j}) = (\lambda p + \frac{1}{\lambda}(1-p))^n$ and

$$\mathbb{E}(Y_n^2) = \prod_{j=1}^n \mathbb{E}(\lambda^{2X_j}) = (\lambda^2 p + \frac{1}{\lambda^2}(1-p))^n$$

d) Using the analysis done in question b), we obtain that

$$\sup_{n\in\mathbb{N}} \mathbb{E}(|Y_n|) < +\infty \quad \text{if and only if} \quad \lambda p + \frac{1}{\lambda}(1-p) \leq 1 \quad \text{if and only if} \quad \lambda \in [1, \frac{1-p}{p}]$$

Likewise,

$$\sup_{n\in\mathbb{N}} \mathbb{E}(Y_n^2) < +\infty \quad \text{if and only if} \quad \lambda^2 p + \frac{1}{\lambda^2}(1-p) \leq 1 \quad \text{if and only if} \quad \lambda \in [1, \sqrt{\frac{1-p}{p}}]$$

as we simply need to replace λ by λ^2 here.