Solutions to Homework 13

Exercise 1*. a) No. H_j is $\mathcal{F}_{j-1}$-measurable, while X_j is independent of $\mathcal{F}_{j-1}$, so

$$\mathbb{E}(H_j X_j) = \mathbb{E}(H_j)\,\mathbb{E}(X_j) = \mathbb{E}(H_j)\,0 = 0$$

b) We have

$$\begin{aligned} A_{n+1} - A_n &= \mathbb{E}(G_{n+1}^2|\mathcal{F}_n) - G_n^2 = \mathbb{E}((G_n + H_{n+1}X_{n+1})^2|\mathcal{F}_n) - G_n^2 \\ &= G_n^2 + 2G_n H_{n+1}\mathbb{E}(X_{n+1}) + H_{n+1}^2\mathbb{E}(X_{n+1}^2) - G_n^2 = H_{n+1}^2 \end{aligned}$$

so $A_0 = 0$ and $A_n = \sum_{j=1}^n H_j^2$.

c) $H_j^2 = 1$ for all j, so $A_n = n$.

d) Here, the idea is to use the optional stopping theorem with the martingale $(G_n^2 - n,\, n \in \mathbb{N})$, which gives

$$\mathbb{E}(G_T^2 - T) = \mathbb{E}(G_0^2 - 0) = 0, \quad \text{so} \quad \mathbb{E}(T) = \mathbb{E}(G_T^2) = a^2$$

Unfortunately, a full justification of the use of the theorem is impossible here using only the tools that you have learned in class, because the martingale $(G_n^2 - n,\, n \in \mathbb{N})$ is not bounded from below until the (unbounded) time T.

Exercise 2. a) Let $(\mathcal{F}_n,\, n \geq 1)$ denote natural filtration of $(X_n, n \in \mathbb{N})$. Using the hint, we find

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = p\,(X_n^2 + 1) + (1-p)\,\frac{X_n}{2} \geq 2p\,X_n + (1-p)\,\frac{X_n}{2} = \frac{3p+1}{2}\,X_n \geq X_n$$

if $p \geq 1/3$. This condition turns out to be the minimal one. Indeed, it can always happen that X_n gets arbitrarily close to the value 1. In this case,

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = p\,(X_n^2 + 1) + (1-p)\,\frac{X_n}{2} \sim 2p + \frac{1-p}{2} = \frac{3p+1}{2}$$

which is strictly less than 1 if $p < \frac{1}{3}$.

b) When $p \geq \frac{1}{3}$, we have

$$\mathbb{E}(X_{n+1}) = \mathbb{E}(\mathbb{E}(X_{n+1}|\mathcal{F}_n)) \geq \mathbb{E}\left(\frac{3p+1}{2}\,X_n\right) = \frac{3p+1}{2}\,\mathbb{E}(X_n)$$

so

$$\mathbb{E}(X_n) \geq \left(\frac{3p+1}{2}\right)^n x$$

c) No. The justification for this is the following: it can always happen that X_n follows the "down" path so as to get arbitrarily close to the value zero. In this case,

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = p\,(X_n^2 + 1) + (1-p)\,\frac{X_n}{2} \sim p > X_n$$

so the supermartingale property (and therefore also the martingale property) fails to hold, for any fixed value of $p > 0$.

Exercise 3. a) Let us compute the function $\psi(m) = \mathbb{E}(|m+U|) = \frac{1}{2}\int_{-1}^{1} |m+u|\, du$. If $m \geq 1$, then

$$\psi(m) = \mathbb{E}(m+U) = m$$

while if $0 \leq m < 1$, then

$$\psi(m) = \frac{1}{2}\left(\int_{-1}^{-m} (-m-u)\, du + \int_{-m}^{+1} (m+u)\, du\right) = \frac{1+m^2}{2}$$

b) By what was computed in part a), we see that for $m \geq 0$, $\psi(m) \geq m$, so

$$\mathbb{E}(M_{n+1} \,|\, \mathcal{F}_n) = \mathbb{E}(|M_n + U_{n+1}| \,|\, \mathcal{F}_n) = \psi(M_n) \geq M_n$$

c)

$$A_{n+1} - A_n = \mathbb{E}(M_{n+1} - M_n | \mathcal{F}_n) = \psi(M_n) - M_n = \begin{cases} 0 & \text{if } M_n \geq 1 \\ \frac{1+M_n^2}{2} - M_n = \frac{(1-M_n)^2}{2} & \text{if } M_n < 1 \end{cases}$$

so $A_n = \frac{1}{2}\sum_{j=1}^{n-1} (1-M_j)^2\, 1_{\{M_j<1\}}$.

d) Yes:

$$\mathbb{E}(M_{n+1}^2 \,|\, \mathcal{F}_n) = \mathbb{E}((M_n + U_{n+1})^2 | \mathcal{F}_n) = M_n^2 + 2M_n \mathbb{E}(U_{n+1}) + \mathbb{E}(U_{n+1}^2) = M_n^2 + \frac{1}{3} \geq M_n^2$$

e) By the previous computation, we deduce that $c = \frac{1}{3}$.

f) No. M is a positive submartingale behaving like a random walk with uniform increments on the positive axis, and being reflected when it gets close to 0. It will not converge anywhere.