Homework 13

Exercise 1. Let $0 < p < 1$ and $x > 0$ be fixed real numbers and $(X_n, n \in \mathbb{N})$ be the process defined recursively as

$$X_0 = x, \quad X_{n+1} = \begin{cases} X_n^2 + 1 & \text{with probability } p \\ X_n/2 & \text{with probability } 1 - p \end{cases} \quad \text{for } n \in \mathbb{N}$$

(a) What minimal condition on $0 < p < 1$ guarantees that the process $X$ is a submartingale (with respect to its natural filtration)? Justify your answer.

*Hint:* The inequality $a^2 + b^2 \geq 2ab$ may be useful here.

(b) For the values of $p$ respecting the condition found in part a), derive a lower bound on $\mathbb{E}(X_n)$.

*Hint:* Proceed recursively.

(c) Does there exist a value of $0 < p < 1$ such that the process $X$ is a martingale? a supermartingale? Again, justify your answer.

Exercise 2*. Let $0 < p < 1$ and $M = (M_n, n \in \mathbb{N})$ be the process defined recursively as

$$M_0 = x \in ]0, 1[, \quad M_{n+1} = \begin{cases} p M_n, & \text{with probability } 1 - M_n \\ (1 - p) + p M_n, & \text{with probability } M_n \end{cases}$$

and $(\mathcal{F}_n, n \in \mathbb{N})$ be the filtration defined as $\mathcal{F}_n = \sigma(M_0, \ldots, M_n), n \in \mathbb{N}$.

(a) For what value(s) of $0 < p < 1$ is the process $M$ a martingale with respect to $(\mathcal{F}_n, n \in \mathbb{N})$? Justify your answer.

(b) In the case(s) $M$ is a martingale, compute $\mathbb{E}(M_{n+1} (1 - M_{n+1}) | \mathcal{F}_n)$ for $n \in \mathbb{N}$.

(c) Deduce the value of $\mathbb{E}(M_n (1 - M_n))$ for $n \in \mathbb{N}$.

(d) Does there exist a random variable $M_\infty$ such that

(i) $M_n \overset{n \to \infty}{\to} M_\infty$ a.s. ?

(ii) $M_n \overset{L^2}{n \to \infty}{\to} M_\infty$ ?

(iii) $\mathbb{E}(M_\infty | \mathcal{F}_n) = M_n, \forall n \in \mathbb{N}$?

(e) What can you say more about $M_\infty$? (No formal justification required here; an intuitive argument will do.)
Exercise 3. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(\mathcal{G}\) be a sub-\(\sigma\)-field of \(\mathcal{F}\). Let \(U \sim \mathcal{U}([-1, +1])\) be a random variable independent of \(\mathcal{G}\) and \(M\) be a positive, integrable and \(\mathcal{G}\)-measurable random variable.

a) Compute the function \(\psi : \mathbb{R}_+ \rightarrow \mathbb{R}\) satisfying

\[
\psi(M) = \mathbb{E}(|M + U| \mid \mathcal{G})
\]

Let now \((U_n, n \geq 1)\) be a sequence of i.i.d.\(\sim \mathcal{U}([-1, +1])\) random variables, all defined on \((\Omega, \mathcal{F}, \mathbb{P})\). Let \(\mathcal{F}_0 = \{\emptyset, \Omega\}\) and \(\mathcal{F}_n = \sigma(U_1, \ldots, U_n), n \geq 1\). Let finally \((M_n, n \geq 1)\) be the process defined recursively as

\[
M_0 = 0, \quad M_{n+1} = |M_n + U_{n+1}|, \quad n \in \mathbb{N}
\]

b) Show that the process \((M_n, n \in \mathbb{N})\) is a submartingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\).

c) Compute the unique predictable and increasing process \((A_n, n \in \mathbb{N})\) such that the process \((M_n - A_n, n \in \mathbb{N})\) is a martingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\).

d) Is it true that the process \((M^2_n, n \in \mathbb{N})\) is also a submartingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\)? Justify your answer.

e) Determine the value of \(c > 0\) such that the process \((N_n = M^2_n - cn, n \in \mathbb{N})\) is a martingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\).

f) Does there exist a random variable \(M_\infty\) such that \(M_n \rightarrow_{n \rightarrow \infty} M_\infty\) almost surely? (Again, no formal justification required here; an intuitive argument will do.)