Solutions to Homework 6

Exercise 1. a) Let us compute for $\varepsilon > 0$:

$$\mathbb{P}(\{|Y_n - 0| > \varepsilon\}) \leq \mathbb{P}(\{Y_n > 0\}) = \mathbb{P}(\{Y_n = 1\}) = \prod_{j=1}^{n} \mathbb{P}(\{X_j = 1\})$$

$$= \prod_{j=1}^{n} \left(1 - \frac{1}{(j+1)^\alpha}\right) \simeq \exp\left(-\sum_{j=1}^{n} \frac{1}{(j+1)^\alpha}\right)$$

where the hint was used in the last (approximate) equality. If $\alpha > 1$, then $\sum_{j=1}^{n} \frac{1}{(j+1)^\alpha}$ converges to a fixed value $< +\infty$ as $n \to \infty$, so $\mathbb{P}(\{Y_n > 0\})$ does not converge to 0 as $n \to \infty$.

On the contrary, if $0 < \alpha \leq 1$, then $\sum_{j=1}^{n} \frac{1}{(j+1)^\alpha} \underset{n\to\infty}{\to} +\infty$, in which case $\mathbb{P}(\{Y_n > 0\}) \underset{n\to\infty}{\to} 0$, so $Y_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$ in this case.

b) The answer is yes. Indeed, we have $\mathbb{E}((Y_n - 0)^2) = \mathbb{E}(Y_n^2) = \mathbb{P}(\{Y_n = 1\})$, so $Y_n \underset{n\to\infty}{\overset{L^2}{\to}} 0$ if and only if $Y_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$.

c) The answer is again yes. Indeed, if for a given realization ω, $Y_n(\omega) = 0$, then $Y_m(\omega) = 0$ for every $m \geq n$, and therefore $\lim_{n\to\infty} Y_n(\omega) = 0$. This implies that

$$\mathbb{P}(\{\lim_{n\to\infty} Y_n = 0\}) \geq \mathbb{P}(\{Y_n = 0\})$$

for any fixed value of $n \geq 1$. If $0 < \alpha \leq 1$, we have seen in question a) that $\mathbb{P}(\{Y_n = 0\}) \underset{n\to\infty}{\to} 1$. So the above inequality implies that $Y_n \underset{n\to\infty}{\to} 0$ almost surely in this case.

Remark. Please note finally that when $\alpha > 1$, convergence in probability does not hold, so automatically in this case, quadratic convergence and almost sure convergence do not hold either.

Exercise 2. a) By independence, we obtain

$$\mathbb{P}\left(\bigcap_{n\geq 1} A_n^c\right) = \prod_{n\geq 1} \mathbb{P}(A_n^c) = \prod_{n\geq 1} (1 - \mathbb{P}(A_n)) \leq \prod_{n\geq 1} \exp(-\mathbb{P}(A_n)) = \exp\left(-\sum_{n\geq 1} \mathbb{P}(A_n)\right) = 0$$

where we have used the fact that $1 - x \leq \exp(-x)$ for $0 \leq x \leq 1$. Therefore, $\mathbb{P}\left(\bigcup_{n\geq 1} A_n\right) = 1$.

Note: The first equality above is "obviously true", but actually needs a proof (not required in the homework): if $(A_n, n \geq 1)$ is a countable sequence of independent events, then it holds that $\mathbb{P}(\cap_{n\geq 1} A_n) = \prod_{n\geq 1} \mathbb{P}(A_n)$. Here is why: define $B_n = \cap_{k=1}^{n} A_k$. Observe that $\cap_{n\geq 1} A_n = \cap_{n\geq 1} B_n$ and $B_n \supset B_{n+1}$ for every $n \geq 1$, so by the continuity property of $\mathbb{P}$,

$$\mathbb{P}(\cap_{n\geq 1} A_n) = \mathbb{P}(\cap_{n\geq 1} B_n) = \lim_{n\to\infty} \mathbb{P}(B_n) = \lim_{n\to\infty} \prod_{k=1}^{n} \mathbb{P}(A_k) = \prod_{n\geq 1} \mathbb{P}(A_n)$$

b) By exactly the same argument as above, we can prove $\mathbb{P}\left(\bigcap_{n\geq N} A_n^c\right) = 0$, $\forall N \geq 1$, and we have seen in class that this holds true if and only if $\mathbb{P}\left(\bigcup_{N\geq 1}\bigcap_{n\geq N} A_n^c\right) = 0$, i.e. $\mathbb{P}\left(\bigcap_{N\geq 1}\bigcup_{n\geq N} A_n\right) = 1$.

c) If for some $\varepsilon > 0$, $\sum_{n\geq 1} \mathbb{P}(\{|X_n| \geq \varepsilon\}) = +\infty$, then by part b), $\mathbb{P}(\{|X_n| \geq \varepsilon \text{ infinitely often}\}) = 1$. This says that almost sure convergence (towards the limiting value 0) of the sequence X_n does not hold, as for this convergence to hold, we would need exactly the opposite, namely that for every $\varepsilon > 0$, $\mathbb{P}(\{|X_n| \geq \varepsilon \text{ infinitely often}\}) = 0$.

d1) For any fixed $\varepsilon > 0$, $\mathbb{P}(\{|X_n| \geq \varepsilon\}) = p_n$ for sufficiently large n, so the minimal condition ensuring convergence in probability is simply $p_n \underset{n\to\infty}{\to} 0$ (said otherwise, $p_n = o(1)$).

d2) $\mathbb{E}((X_n - 0)^2) = n^2\, p_n$, so the minimal condition for L^2 convergence is $p_n = o(\frac{1}{n^2})$.

d3) Using the two Borel-Cantelli lemmas (both applicable here as the X_n are independent), we see that the minimal condition for almost sure convergence is $\sum_{n\geq 1} p_n < +\infty$, satisfied in particular if $p_n = O(n^{-1-\delta})$.

e1) We have in this case, for any fixed $\varepsilon > 0$:

$$\mathbb{P}(\{|Y_n| \geq \varepsilon\}) = 2\int_\varepsilon^{+\infty} dx\, \frac{1}{\pi}\,\frac{\lambda_n}{\lambda_n^2 + x^2} = \frac{2}{\pi}\left(\frac{\pi}{2} - \arctan\left(\frac{\varepsilon}{\lambda_n}\right)\right) \underset{n\to 0}{\to} 0$$

if and only if $\lambda_n \underset{n\to\infty}{\to} 0$.

e2) $\mathbb{E}(Y_n^2) = +\infty$ in all cases, so L^2 convergence does not hold.

e3) Observe first that by the change of variable $y = \lambda_n x$,

$$\mathbb{P}(\{|Y_n| \geq \varepsilon\}) = 2\int_\varepsilon^{+\infty} dy\, \frac{\lambda_n}{\pi\,(\lambda_n^2 + y^2)} = 2\int_{\varepsilon/\lambda_n}^{+\infty} dx\, \frac{1}{\pi(1+x^2)} \simeq 2\int_{\varepsilon/\lambda_n}^{+\infty} dx\, \frac{\lambda_n}{\pi x^2} = \frac{2\lambda_n}{\pi\,\varepsilon}$$

when λ_n is small. So the minimal condition for almost sure convergence is $\sum_{n\geq 1} \lambda_n < +\infty$, satisfied in particular if $\lambda_n = O(n^{-1-\delta})$.

Exercise 3. a) For a given $\varepsilon > 0$, let us first consider n sufficiently large such that

$$\left|\frac{\mu_1 + \ldots + \mu_n}{n} - \mu\right| < \frac{\varepsilon}{2}$$

(such an n exists by assumption). For the same value of n, we have

$$\begin{aligned}
\mathbb{P}\left(\left\{\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right\}\right) &\leq \mathbb{P}\left(\left\{\left|\frac{S_n}{n} - \frac{\mu_1 + \ldots + \mu_n}{n}\right| \geq \frac{\varepsilon}{2}\right\}\right) \\
&= \mathbb{P}\left(\left\{\left|\sum_{i=1}^n (X_i - \mu_i)\right| \geq \frac{n\varepsilon}{2}\right\}\right) \leq \frac{4}{n^2\,\varepsilon^2}\,\mathbb{E}\left(\left(\sum_{i=1}^n (X_i - \mu_i)\right)^2\right) \\
&= \frac{4}{n^2\,\varepsilon^2}\sum_{i,j=1}^n \mathrm{Cov}(X_i, X_j) \leq \frac{4C_1}{n^2\,\varepsilon^2}\sum_{i,j=1}^n \exp(-C_2\,|i-j|) \\
&\leq \frac{8C_1}{n\,\varepsilon^2}\sum_{k\in\mathbb{Z}} \exp(-C_2\,|k|) \underset{n\to\infty}{\longrightarrow} 0, \quad \text{as} \quad \sum_{k\in\mathbb{Z}} \exp(-C_2\,|k|) < +\infty
\end{aligned}$$

b) The strong law does hold in this case. However, it is not clear if the particular proof we did in class could be made to work. Our proof breaks down at the last step where the sequence $\{X_n, n \geq 1\}$ is decomposed into two sequences $\{X_n^+, n \geq 1\}$ and $\{X_n^-, n \geq 1\}$. In this case, just because the original sequence satisfies the assumption in the problem, does not mean that the positive part and the negative part would each satisfy the same assumptions. The assumption on the mean is easy enough to check. The difficult part seems to be the weak correlation assumption.

That being said, it is possible to modify a different proof of the strong law of large numbers which does not rely on the decomposition into positive and negative parts.

c) We can check here that for $n \geq m \geq 1$, we have

$$\operatorname{Cov}(X_n, X_m) = a^{n-m} \operatorname{Var}(X_m)$$

and also that $\operatorname{Var}(X_1) = 0$ and

$$\operatorname{Var}(X_m) = 1 + a^2 \operatorname{Var}(X_{m-1}) = \ldots = 1 + a^2 + a^4 + \ldots + a^{2(m-2)} \quad \text{for } m \geq 2$$

From this, we conclude that when $|a| < 1$, $\operatorname{Cov}(X_n, X_m)$ satisfies the condition given in the problem set. Besides, for every $n \geq 1$, we have

$$\mu_n = \mathbb{E}(X_n) = a\,\mathbb{E}(X_{n-1}) = a^{n-1}\,x$$

so

$$\lim_{n\to\infty} \frac{\mu_1 + \ldots + \mu_n}{n} = \frac{1}{n} \sum_{j=1}^{n} a^{j-1}\,x \underset{n\to\infty}{\to} 0$$

when $|a| < 1$, for any value of $x \in \mathbb{R}$. So $\mu = 0$ in this case and

$$\frac{S_n}{n} = \frac{X_1 + \ldots + X_n}{n} \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$$

Exercise 4. a) For $\varepsilon > 0$ and $n \geq 1$ fixed, let us compute, using the law of total probability:

$$\begin{aligned}
\mathbb{P}\left(\left\{\left|\frac{X_1 + \ldots + X_{T_n}}{T_n} - \mu\right| \geq \varepsilon\right\}\right) &= \sum_{k\geq 1} \mathbb{P}\left(\left\{\left|\frac{X_1 + \ldots + X_{T_n}}{T_n} - \mu\right| \geq \varepsilon\right\} \,\middle|\, \{T_n = k\}\right) \cdot \mathbb{P}(\{T_n = k\}) \\
&= \sum_{k\geq 1} \mathbb{P}\left(\left\{\left|\frac{X_1 + \ldots + X_k}{k} - \mu\right| \geq \varepsilon\right\} \,\middle|\, \{T_n = k\}\right) \cdot \mathbb{P}(\{T_n = k\}) \\
&= \sum_{k\geq 1} \mathbb{P}\left(\left\{\left|\frac{X_1 + \ldots + X_k}{k} - \mu\right| \geq \varepsilon\right\}\right) \cdot p_k^{(n)}
\end{aligned}$$

by independence of T_n and the sequence $(X_n,\, n \geq 1)$. From the proof of the weak law of large numbers, we know that for every $k \geq 1$:

$$\mathbb{P}\left(\left\{\left|\frac{X_1 + \ldots + X_k}{k} - \mu\right| \geq \varepsilon\right\}\right) \leq \frac{\sigma^2}{k\,\varepsilon^2}$$

so

$$\mathbb{P}\left(\left\{\left|\frac{X_1 + \ldots + X_{T_n}}{T_n} - \mu\right| \geq \varepsilon\right\}\right) \leq \frac{\sigma^2}{\varepsilon^2} \sum_{k\geq 1} \frac{p_k^{(n)}}{k}$$

A sufficient condition ensuring convergence in probability is therefore: $\lim_{n\to\infty} \sum_{k\geq 1} \frac{p_k^{(n)}}{k} = 0$.

b1) Let us compute for $n \geq 1$ and $k \geq 2$: (noting that the probability is equal to zero for $k = 1$)

$$\begin{aligned} p_k^{(n)} &= \mathbb{P}(\{T_n = k\}) = \sum_{j=1}^{k-1} \mathbb{P}(\{G_{n1} = j,\, T_n = k\}) = \sum_{j=1}^{k-1} \mathbb{P}(\{G_{n1} = j,\, G_{n2} = k - j\}) \\ &= \sum_{j=1}^{k-1} \mathbb{P}(\{G_{n1} = j\}) \cdot \mathbb{P}(\{G_{n2} = k - j\}) = \sum_{j=1}^{k-1} q_n^{j-1}\,(1 - q_n)\, q_n^{k-j-1}\,(1 - q_n) \\ &= (k-1)\, q_n^{k-2}\,(1 - q_n)^2 \end{aligned}$$

This implies that

$$\begin{aligned} \mathbb{E}(T_n) &= \sum_{k\geq 2} k\,(k-1)\, q_n^{k-2}\,(1 - q_n)^2 = \frac{\partial^2}{\partial z^2} \left(\sum_{k\geq 2} z^k \right) \Big|_{z=q_n} (1 - q_n)^2 \\ &= \frac{\partial^2}{\partial z^2} \left(\frac{1}{1-z} - 1 - z \right) \Big|_{z=q_n} (1 - q_n)^2 = \frac{2}{(1-q_n)^3}\,(1 - q_n)^2 = \frac{2}{1 - q_n} \end{aligned}$$

Note: This result could also have been obtained using $\mathbb{E}(T_n) = \mathbb{E}(G_{n1}) + \mathbb{E}(G_{n2})$ together with the fact that a geometric random variable with parameter q has expectation $1/(1-q)$. [NB: geometric random variables with parameter q can be defined either on $\mathbb{N}^* = \{1, 2, 3, \ldots\}$ (as it is the case here) or on $\mathbb{N} = \{0, 1, 2, \ldots\}$, as it was the case in Ex. 3 of Homework 4; their expectation is equal to $q/(1-q)$ in the latter case]

b2) From the above computations, we see that

$$\sum_{k\geq 1} \frac{p_k^{(n)}}{k} = \sum_{k\geq 2} \frac{k-1}{k}\, q_n^{k-2}\,(1 - q_n)^2 \leq \sum_{k\geq 2} q_n^{k-2}\,(1 - q_n)^2 = \frac{1}{1 - q_n}\,(1 - q_n)^2 = 1 - q_n$$

so convergence in probability occurs if $q_n \underset{n\to\infty}{\to} 1$. This is in accordance with the fact that $\mathbb{E}(T_n) \underset{n\to\infty}{\to} +\infty$ in this case (see part a).