Problem 1

question a *(5 points)*

(2 points) Hilbert Space: $\mathbb{C}^4$ or $\mathbb{C}^2 \otimes \mathbb{C}^2$

(3 points) Unitary Matrix (be careful for the order)

$$U = (S_1 \otimes S_2)(R_1 \otimes R_2)(D_1(\alpha) \otimes D_2(\beta)) \tag{1}$$

question b *(7 points)*

(2 points) First dephasing steps:

$$|\psi_1\rangle = (D(\alpha) \otimes D(\beta)) |\psi\rangle = \frac{1}{\sqrt{2}} \left(e^{i(\alpha+\beta)} |xx\rangle + |yy\rangle \right) \tag{2}$$

(2 points) Then the semi-reflecting mirrors:

$$|\psi_2\rangle = (R \otimes R) |\psi_1\rangle = \frac{1}{\sqrt{2}} \left(e^{i(\alpha+\beta)} |yy\rangle + |xx\rangle \right) \tag{3}$$

(3 points) then the semi-transparents mirrors:

$$|\psi_3\rangle = (S \otimes S) |\psi_2\rangle = \frac{1}{2\sqrt{2}} \left(e^{i(\alpha+\beta)} (|x\rangle_1 - |y\rangle_1) \otimes (|x\rangle_2 - |y\rangle_2) \right) \tag{4}$$

$$+ \frac{1}{2\sqrt{2}} \left((|x\rangle_1 + |y\rangle_1) \otimes (|x\rangle_2 + |y\rangle_2) \right) \tag{5}$$

$$= \frac{1}{2\sqrt{2}} \left((1 + e^{i(\alpha+\beta)})(|xx\rangle + |yy\rangle) + (1 - e^{i(\alpha+\beta)})(|xy\rangle + |yx\rangle) \right) \tag{6}$$

question c *(3 points)*

(2 points) Possible states after measurement: $\{|xx\rangle, |yy\rangle, |xy\rangle, |yx\rangle\}$

(1 points) 2 detectors will always click at each shot

question d *(4 points)*

$$P(|\psi\rangle \to |yy\rangle) = P(|\psi\rangle \to |xx\rangle) = \frac{1}{2} \cos\left(\frac{\alpha + \beta}{2} \right)^2 \tag{7}$$

$$P(|\psi\rangle \to |xy\rangle) = P(|\psi\rangle \to |yx\rangle) = \frac{1}{2} \sin\left(\frac{\alpha + \beta}{2} \right)^2 \tag{8}$$

$$\tag{9}$$

Other possible solution:

$$P(|\psi\rangle \to |yy\rangle) = P(|\psi\rangle \to |xx\rangle) = \frac{1}{4}\left(1 + \cos(\alpha+\beta)\right) \tag{10}$$
$$P(|\psi\rangle \to |xy\rangle) = P(|\psi\rangle \to |yx\rangle) = \frac{1}{4}\left(1 - \cos(\alpha+\beta)\right) \tag{11}$$
$$\tag{12}$$

question e *(6 points)*

(2 points) Same direction with probability 1:

$$P(|\psi\rangle \to |xy\rangle) = P(|\psi\rangle \to |yx\rangle) = \frac{1}{2}\sin\left(\frac{\alpha+\beta}{2}\right)^2 = 0 \tag{13}$$
$$\iff \frac{\alpha+\beta}{2} \in \pi\mathbb{Z} \tag{14}$$
$$\iff (\alpha+\beta) \in 2\pi\mathbb{Z} \tag{15}$$

(2 points) Opposite direction with probability 1:

$$P(|\psi\rangle \to |yy\rangle) = P(|\psi\rangle \to |xx\rangle) = \frac{1}{2}\cos\left(\frac{\alpha+\beta}{2}\right)^2 = 0 \tag{16}$$
$$\iff \frac{\alpha+\beta}{2} \in \frac{\pi}{2} + \pi\mathbb{Z} \tag{17}$$
$$\iff (\alpha+\beta) \in \pi + 2\pi\mathbb{Z} \tag{18}$$

(2 points) any direction with uniform probability:

$$\cos\left(\frac{\alpha+\beta}{2}\right)^2 = \sin\left(\frac{\alpha+\beta}{2}\right)^2 \tag{19}$$
$$\iff \frac{\alpha+\beta}{2} \in \frac{\pi}{4} + \frac{\pi}{2}\mathbb{Z} \tag{20}$$
$$\iff (\alpha+\beta) \in \frac{\pi}{2} + \pi\mathbb{Z} \tag{21}$$

Problem 2

question a *(6 points)*

(4 points) Calculus:

$$\begin{aligned} P(a_k = 1, b_k = 1) &= P(|B\rangle \to |\alpha_k\rangle \otimes |\beta_k\rangle) && (22)\\ &= |\langle B|\alpha_k, \beta_k\rangle|^2 && (23)\\ &= \frac{1}{2}\left(\langle\alpha_k|0\rangle\langle\beta_k|0\rangle + \langle\alpha_k|1\rangle\langle\beta_k|1\rangle\right)^2 && (24)\\ &= \frac{1}{2}\left(\cos(\alpha_k)\cos(\beta_k) + \sin(\alpha_k)\sin(\beta_k)\right)^2 && (25)\\ &= \frac{1}{2}\cos(\alpha_k - \beta_k)^2 && (26) \end{aligned}$$

$$\begin{aligned} P(a_k = 1, b_k = -1) &= P(|B\rangle \to |\alpha_k\rangle \otimes |\beta_{k,\perp}\rangle) && (27)\\ &= |\langle B|\alpha_k, \beta_{k,\perp}\rangle|^2 && (28)\\ &= \frac{1}{2}\left(\langle\alpha_k|0\rangle\langle\beta_{k,\perp}|0\rangle + \langle\alpha_k|1\rangle\langle\beta_{k,\perp}|1\rangle\right)^2 && (29)\\ &= \frac{1}{2}\left(-\cos(\alpha_k)\sin(\beta_k) + \sin(\alpha_k)\cos(\beta_k)\right)^2 && (30)\\ &= \frac{1}{2}\sin(\alpha_k - \beta_k)^2 && (31) \end{aligned}$$

Similarly:

$$\begin{aligned} P(a_k = -1, b_k = -1) &= \frac{1}{2}\cos(\alpha_k - \beta_k)^2 && (32)\\ P(a_k = -1, b_k = 1) &= \frac{1}{2}\sin(\alpha_k - \beta_k)^2 && (33)\\ & && (34) \end{aligned}$$

(2 points) Calculus:

$$P(a_k = 1) = P(a_k = 1, b_k = 1) + P(a_k = 1, b_k = -1) = \frac{1}{2} \quad (35)$$

$$P(a_k = -1) = P(a_k = -1, b_k = 1) + P(a_k = -1, b_k = -1) = \frac{1}{2} \quad (36)$$

$$P(b_k = 1) = P(a_k = 1, b_k = 1) + P(a_k = -1, b_k = 1) = \frac{1}{2} \quad (37)$$

$$P(b_k = -1) = P(a_k = 1, b_k = -1) + P(a_k = -1, b_k = -1) = \frac{1}{2} \quad (38)$$

question b *(3 points)*

(2 points) When $\alpha_k = \beta_k = 0$ we have:

$$P(a_k = -1, b_k = 1) = P(a_k = 1, b_k = -1) = \frac{1}{2}\sin(0)^2 = 0 \tag{39}$$

Or similarly:

$$P(a_k = -1, b_k = 1) = P(a_k = 1, b_k = -1) = \frac{1}{2}\cos(0)^2 = \frac{1}{2} \tag{40}$$

(1 points) Length Calculus:

$$P(\alpha_k = 0, \beta_k = 0) = P(\alpha_k = 0)P(\beta_k = 0) = \frac{1}{9} \tag{41}$$

So if $L(N)$ is the length:

$$L(N) = \sum_{k=1}^{N} \delta_0(\alpha_k)\delta_0(\beta_k) \tag{42}$$

Thus using the Law of Large Number, when N is large:

$$\frac{L(N)}{N} \simeq \mathbb{E}\left[\delta_0(\alpha_k)\delta_0(\beta_k)\right] = P(\alpha_k = 0, \beta_k = 0) = \frac{1}{9} \tag{43}$$

question c *(5 points)*

(3 points) Empirical average:

$$\frac{1}{\frac{N}{9}}\left(\sum_{\alpha_k=0,\beta_k=\frac{\pi}{8}} a_k b_k + \sum_{\alpha_k=0,\beta_k=\frac{\pi}{8}} a_k b_k + \sum_{\alpha_k=-\frac{\pi}{4},\beta_k=\frac{\pi}{8}} a_k b_k + \sum_{\alpha_k=-\frac{\pi}{4},\beta_k=-\frac{\pi}{8}} a_k b_k\right) \tag{44}$$

(1 points) Theoretical average: $\langle B|\, S\, |B\rangle$

(1 points) Expected result from Bell inequality: $2\sqrt{2}$

question d *(3 points)*

(1 points) Recall the Bell state satisfies the relation for any θ:

$$|B\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\left(|\theta\theta\rangle + |\theta_\perp\theta_\perp\rangle\right) \tag{45}$$

(2 points) Therefore, for any measurement of Eve in the basis $\{|\theta\rangle, |\theta_\perp\rangle\}$, it leaves the particles in the state $|\theta\theta\rangle$ or $|\theta_\perp\theta_\perp\rangle$

question e *(8 points)*

(5 points) In general:

$$\langle\gamma\gamma'| A(\alpha) \otimes B(\beta) |\gamma\gamma'\rangle = \langle\gamma| A(\alpha) |\gamma\rangle \langle\gamma'| B(\alpha) |\gamma'\rangle \tag{46}$$

And:

$$\begin{aligned}
\langle\gamma| A(\alpha) |\gamma\rangle &= (+1 \langle\gamma|\alpha\rangle \langle\alpha|\gamma\rangle + (-1) \langle\gamma|\alpha_\perp\rangle \langle\alpha_\perp|\gamma\rangle) && (47)\\
&= \cos(\alpha-\gamma)^2 - \sin(\alpha-\gamma)^2 && (48)\\
&= \cos(2(\alpha-\gamma)) && (49)
\end{aligned}$$

And similarly:

$$\langle\gamma'| B(\beta) |\gamma'\rangle = \cos(2(\beta-\gamma')) \tag{50}$$

So In general:

$$\langle\gamma\gamma'| A(\alpha) \otimes B(\beta) |\gamma\gamma'\rangle = \cos(2(\alpha-\gamma))\cos(2(\beta-\gamma')) \tag{51}$$

So:

$$\begin{aligned}
\langle\gamma\gamma'| S |\gamma\gamma'\rangle &= \cos(2\gamma)\cos\left(\frac{\pi}{4}-2\gamma'\right) + \cos(2\gamma)\cos\left(\frac{\pi}{4}+2\gamma'\right) && (52)\\
&- \cos\left(\frac{\pi}{2}+2\gamma\right)\cos\left(\frac{\pi}{4}-2\gamma'\right) + \cos\left(\frac{\pi}{2}+2\gamma\right)\cos\left(\frac{\pi}{4}+2\gamma'\right) && (53)
\end{aligned}$$

(1 points) As stated in previous question, we have: $|\gamma\gamma'\rangle = |\theta\theta\rangle$ or $|\gamma\gamma'\rangle = |\theta_\perp\theta_\perp\rangle$

(2 points) For $|\gamma\gamma'\rangle = |\theta\theta\rangle$:

$$\begin{aligned}
\langle\gamma\gamma'| S |\gamma\gamma'\rangle &= \cos(2\theta)\cos\left(\frac{\pi}{4}-2\theta\right) + \cos(2\theta)\cos\left(\frac{\pi}{4}+2\theta\right) && (54)\\
&- \cos\left(\frac{\pi}{2}+2\theta\right)\cos\left(\frac{\pi}{4}-2\theta\right) + \cos\left(\frac{\pi}{2}+2\theta\right)\cos\left(\frac{\pi}{4}+2\theta\right) && (55)\\
&= \cos(2\theta)\left(\cos\left(\frac{\pi}{4}-2\theta\right) + \cos\left(\frac{\pi}{4}+2\theta\right)\right) && (56)\\
&+ \cos\left(\frac{\pi}{2}+2\theta\right)\left(\cos\left(\frac{\pi}{4}+2\theta\right) - \cos\left(\frac{\pi}{4}-2\theta\right)\right) && (57)\\
&= \cos(2\theta)2\cos\left(\frac{\pi}{4}\right)\cos(2\theta) - \cos\left(\frac{\pi}{2}+2\theta\right)2\sin\left(\frac{\pi}{4}\right)\sin(2\theta) && (58)\\
&= \sqrt{2}\left(\cos(2\theta)\cos(2\theta) - \cos\left(\frac{\pi}{2}+2\theta\right)\sin(2\theta)\right) && (59)\\
&= \sqrt{2}\left(\cos(2\theta)\cos(2\theta) + \sin(2\theta)\sin(2\theta)\right) && (60)\\
&= \sqrt{2}\cos(2\theta-2\theta) && (61)\\
&= \sqrt{2} && (62)\\
& && (63)
\end{aligned}$$

So the test would be to check if the empirical correlation is above this value.

Problem 3

question a *(6 points)*

(1 points) Recall: $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$ and $\sigma_x\sigma_y + \sigma_y\sigma_x = 0$ etc

(2 points) Calculus:

$$(\hat{n}\cdot\vec{\sigma})^2 = n_x^2\sigma_x^2 + n_y^2\sigma_y^2 + n_z^2\sigma_z^2 + n_xn_y(\sigma_x\sigma_y+\sigma_y\sigma_x) + n_xn_z(\sigma_x\sigma_z+\sigma_z\sigma_x) + n_yn_z(\sigma_y\sigma_z+\sigma_z\sigma_y) \quad (64)$$

$$= (n_x^2+n_y^2+n_z^2)I = I \quad (65)$$

(3 points) Calculus:

$$e^{i\alpha\hat{n}\cdot\vec{\sigma}} = \sum_{k\geq 0}\frac{1}{k!}i^k\alpha^k(\hat{n}\cdot\vec{\sigma})^k \quad (66)$$

$$= \sum_{k\geq 0}\frac{1}{(2k)!}i^{2k}\alpha^{2k}I + \sum_{k\geq 0}\frac{1}{(2k+1)!}i^{2k+1}\alpha^{2k+1}\hat{n}\cdot\vec{\sigma} \quad (67)$$

$$= \sum_{k\geq 0}\frac{1}{(2k)!}(-1)^k\alpha^{2k}I + i\sum_{k\geq 0}\frac{1}{(2k+1)!}(-1)^k\alpha^{2k+1}\hat{n}\cdot\vec{\sigma} \quad (68)$$

$$= \cos(\alpha)I + i\sin(\alpha)\hat{n}\cdot\vec{\sigma} \quad (69)$$

$$(70)$$

question b *(6 points)*

(2 points) The representation:

$$U(t) = \exp\left(\frac{-it}{\hbar}\left(\frac{-\hbar\Delta}{2}\sigma_z + \frac{-\hbar\omega_1}{2}\sigma_x\right)\right) \quad (71)$$

$$= \exp\left(i\underbrace{\frac{t\sqrt{\omega_1^2+\Delta^2}}{2}}_{\alpha}\underbrace{\left(\frac{\omega_1}{\sqrt{\omega_1^2+\Delta^2}}\sigma_x + \frac{\Delta}{\sqrt{\omega_1^2+\Delta^2}}\sigma_z\right)}_{\hat{n}\cdot\sigma}\right) \quad (72)$$

(2 points) The solution to the first bullet point is the first column of the matrix:

$$U(t) = \begin{pmatrix} \cos(\alpha) + in_z\sin(\alpha) & in_x\sin(\alpha) \\ in_x\sin(\alpha) & \cos(\alpha) - in_z\sin(\alpha) \end{pmatrix} \quad (73)$$

(2 points) for the second bullet point:

$$\alpha \simeq \frac{t_1}{2}\omega_1 = \frac{\pi}{4} \qquad n_x \simeq 1 \qquad n_z = \frac{\Delta}{\omega_1}\left(1 - \frac{1}{2}\frac{\Delta^2}{\omega_1^2}(1+o(1))\right) = o(1) \quad (74)$$

So the vector:

$$|\psi(t_1)\rangle \simeq \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \tag{75}$$

question c *(6 points)*

Assuming this is the opposite: $\frac{\omega_1}{\Delta} \simeq 0$

(2 points) Approximation:

$$\alpha \simeq \frac{T}{2}\Delta = \frac{\pi}{2} \qquad n_x = o(1) \qquad n_z \simeq 1 \tag{76}$$

(2 points) The matrix:

$$U(t_1, t_1 + T) \simeq \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{77}$$

(2 points) the vector:

$$|\psi(t_1 + T)\rangle = U(t_1, t_1 + T)\,|\psi(t_1)\rangle \simeq \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{78}$$

question d *(4 points)*

(2 points) Matrix:

$$U(t_1 + T, 2t_1 + T) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{79}$$

(2 points) Vector:

$$|\psi(2t_1 + T)\rangle = U(t_1 + T, 2t_1 + T)\,|\psi(t_1 + T)\rangle \simeq \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} i \\ 0 \end{pmatrix} = e^{i\frac{\pi}{2}}\,|\psi(0)\rangle \tag{80}$$

question e *(3 points)*

(Check Notes)