Final Exam Solution
Traitement Quantique de l'Information

Problem 1: Quiz *20 points*

1) *[5 pts]* True. Since $\mathbf{M}$ is a unitary matrix ($\mathbf{M}\mathbf{M}^\dagger = \mathbf{M}^\dagger\mathbf{M} = I$), it is a valid quantum operation.

2) *[5 pts]* False. Since the four states are orthogonal in $\mathbb{C}^2 \otimes \mathbb{C}^2$, so they can be cloned.

3) *[5 pts]* False. $|\uparrow\rangle$ and $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ are not orthogonal in $\mathbb{C}^2$, but they are represented by orthogonal vectors on Bloch sphere.

4) *[5 pts]* False. Since Bob is measuring in the $\{|\alpha\rangle, |\alpha_\perp\rangle\}$ basis, his qubit will be in one these two states, independent of Alice's measurement.

Problem 2: Interferometer *30 pts*

1) *[10 pts]*After the first semi-transparent mirror the photon in state

$$\mathbf{H}|2\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$$

After the reflecting mirrors the state is

$$\mathbf{R}\big(\frac{|1\rangle - |2\rangle}{\sqrt{2}}\big) = \frac{-|1\rangle + |2\rangle}{\sqrt{2}}$$

After the dephasor,

$$\mathbf{P}_\phi\big(\frac{-|1\rangle + |2\rangle}{\sqrt{2}}\big) = \frac{-e^{i\phi}|1\rangle + |2\rangle}{\sqrt{2}}$$

And, after the last semi-transparent mirror the state is

$$\begin{aligned}\mathbf{H}\big(\frac{-e^{i\phi}|1\rangle + |2\rangle}{\sqrt{2}}\big) &= \frac{1}{\sqrt{2}}\Big(-\frac{e^{i\phi}}{\sqrt{2}}(|1\rangle + |2\rangle) + \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)\Big)\\ &= \frac{1-e^{i\phi}}{2}|1\rangle - \frac{1+e^{i\phi}}{2}|2\rangle\end{aligned}$$

2) *[5 pts]*

$$\mathbb{P}(D_1) = \Big|\frac{1-e^{i\phi}}{2}\Big|^2 = \frac{1}{4}\big((1-cos(\phi))^2 + sin(\phi)^2\big) = \frac{1}{4}(2-2cos(\phi)) = \frac{1}{2}(1-cos(\phi)) = sin(\frac{\phi}{2})^2$$

$$\mathbb{P}(D_2) = \Big|\frac{1+e^{i\phi}}{2}\Big|^2 = \frac{1}{4}\big((1+cos(\phi))^2 + sin(\phi)^2\big) = \frac{1}{4}(2+2cos(\phi)) = \frac{1}{2}(1+cos(\phi)) = cos(\frac{\phi}{2})^2$$

Either D_1 or D_2 clic. Only one receives energy for each coming photon.

3) *[2 pts]* Matrix $\mathbf{A}$ has to be unitary, $\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger\mathbf{A} = I$. From this constraint we must have that columns (and rows) of the matrix are unit norm and orthogonal.
First and the third columns must be orthogonal, so $\epsilon X = -\epsilon\sqrt{1-\epsilon^2}$, which implies (for $\epsilon \neq 0$) $X = -\sqrt{1-\epsilon^2}$.

4) *[13 pts]* Recall that the state after the dephrasor is $\frac{-e^{i\phi}|1\rangle+|2\rangle}{\sqrt{2}}$. Applying matrix $\mathbf{A}$, we get

$$\begin{aligned}
\mathbf{A}\Big(\frac{-e^{i\phi}|1\rangle+|2\rangle}{\sqrt{2}}\Big) &= \frac{-e^{i\phi}}{\sqrt{2}}\mathbf{A}|1\rangle + \frac{1}{\sqrt{2}}\mathbf{A}|2\rangle \\
&= \frac{-e^{i\phi}}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}(\sqrt{1-\epsilon^2}|0\rangle + \epsilon|2\rangle) \\
&= \sqrt{\frac{1-\epsilon^2}{2}}|0\rangle - \frac{e^{i\phi}}{\sqrt{2}}|1\rangle + \frac{\epsilon}{\sqrt{2}}|2\rangle
\end{aligned}$$

After the last semi-transparent mirror the final state is

$$\begin{aligned}
\mathbf{H}\Big(\sqrt{\frac{1-\epsilon^2}{2}}|0\rangle - \frac{e^{i\phi}}{\sqrt{2}}|1\rangle + \frac{\epsilon}{\sqrt{2}}|2\rangle\Big) &= \sqrt{\frac{1-\epsilon^2}{2}}\mathbf{H}|0\rangle - \frac{e^{i\phi}}{\sqrt{2}}\mathbf{H}|1\rangle + \frac{\epsilon}{\sqrt{2}}\mathbf{H}|2\rangle \\
&= \sqrt{\frac{1-\epsilon^2}{2}}|0\rangle - \frac{e^{i\phi}}{\sqrt{2}}\frac{|1\rangle+|2\rangle}{\sqrt{2}} + \frac{\epsilon}{\sqrt{2}}\frac{|1\rangle-|2\rangle}{\sqrt{2}} \\
&= \sqrt{\frac{1-\epsilon^2}{2}}|0\rangle + \frac{\epsilon-e^{i\phi}}{2}|1\rangle - \frac{\epsilon+e^{i\phi}}{2}|2\rangle
\end{aligned}$$

So, for probabilities we have

$$\mathbb{P}(\text{Absorption}) = \frac{1-\epsilon^2}{2}$$

$$\mathbb{P}(D_1) = |\frac{\epsilon-e^{i\phi}}{2}|^2 = \frac{1}{4}\big((\epsilon - cos(\phi))^2 + sin(\phi)^2\big) = \frac{1}{4}(1+\epsilon^2-2\epsilon cos(\phi))$$

$$\mathbb{P}(D_2) = |\frac{\epsilon+e^{i\phi}}{2}|^2 = \frac{1}{4}\big((\epsilon + cos(\phi))^2 + sin(\phi)^2\big) = \frac{1}{4}(1+\epsilon^2+2\epsilon cos(\phi))$$

We can see that the probabilities sum to 1.

Problem 3: Entanglement *20 pts*

1) *[5 pts]*$\mathcal{H} = \mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$, since there 4 qubits in total. The total state is $|\phi\rangle\otimes|\psi\rangle$

2) *[10 pts]* Possible states of Alice's qubits are $|B_1\rangle, |B_2\rangle, |B_3\rangle, |B_4\rangle$. Suppose they are in state $|B_1\rangle$, so the measurement by Alice is the projection $|B_1\rangle\langle B_1|$ applying on qubits in Alice's possession. Now, we compute the states of Bob and Charlies's qubits.

Firs note that, we have

$$\langle B_1|_A(\alpha|0\rangle + \beta|1\rangle)_A(\frac{1}{\sqrt{2}}|0\rangle_A|00\rangle_{BC} + \frac{1}{\sqrt{2}}|1\rangle_A|11\rangle_{BC})$$
$$= \langle B_1|_A\frac{1}{\sqrt{2}}(\alpha|00\rangle_A|00\rangle_{BC} + \alpha|01\rangle_A|11\rangle_{BC} + \beta|10\rangle_A|00\rangle_{BC} + \beta|11\rangle_A|11\rangle_{BC})$$
$$= \frac{\alpha}{2}|00\rangle_{BC} + \frac{\beta}{2}|11\rangle_{BC}$$

$$|B_1\rangle_A\langle B_1|_A(\alpha|0\rangle + \beta|1\rangle)_A(\frac{1}{\sqrt{2}}|0\rangle_A|00\rangle_{BC} + \frac{1}{\sqrt{2}}|1\rangle_A|11\rangle_{BC}) = |B_1\rangle_A\frac{1}{2}(\alpha|00\rangle_{BC} + \beta|11\rangle_{BC})$$

Normalizing the state, we get

$$|B_1\rangle_A(\alpha|00\rangle_{BC} + \beta|11\rangle_{BC})$$

By similar calculations for other possible states, we find that the final possible states are

$$|B_1\rangle_A(\alpha|00\rangle_{BC} + \beta|11\rangle_{BC})$$
$$|B_2\rangle_A(\alpha|11\rangle_{BC} + \beta|00\rangle_{BC})$$
$$|B_3\rangle_A(\alpha|11\rangle_{BC} - \beta|00\rangle_{BC})$$
$$|B_4\rangle_A(\alpha|00\rangle_{BC} - \beta|11\rangle_{BC})$$

3) *[5 pts]* Only two classical bits are required, one for Bob and one for Charlie.

If Alices measures $|B_1\rangle$, then Bob and Charlie's qubits are in the state $\alpha|00\rangle_{BC} + \beta|11\rangle_{BC}$, so they should do nothing. Alice sends 0 to both Bob and Charlie.

If Alices measures $|B_2\rangle$, then Bob and Charlie's qubits are in the state $\alpha|11\rangle_{BC} + \beta|00\rangle_{BC}$, so they should apply $\mathbf{X}_B \otimes \mathbf{X}_C$. Alice sends 1 to both Bob and Charlie.

If Alices measures $|B_3\rangle$, then Bob and Charlie's qubits are in the state $\alpha|11\rangle_{BC} - \beta|00\rangle_{BC}$, so they should apply $\mathbf{X}_B \otimes \mathbf{X}_C$. Alice sends 1 to both Bob and Charlie.

If Alices measures $|B_4\rangle$, then Bob and Charlie's qubits are in the state $\alpha|00\rangle_{BC} - \beta|11\rangle_{BC}$, so they should do nothing. Alice sends 0 to both Bob and Charlie.

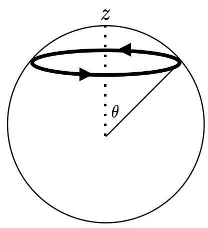
Problem 4: Spin and Density Matrix *30 pts*
1) *[6 pts]*

$$\mathbf{U}_t = e^{-\frac{it}{\hbar}\mathbf{H}} = e^{\frac{i\omega t}{2}\sigma_z} = \begin{bmatrix} e^{\frac{i\omega t}{2}} & 0 \\ 0 & e^{-\frac{i\omega t}{2}} \end{bmatrix}$$

In the last equality, we used the fact that σ_z is a diagonal matrix. Note that $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenvectors of $\mathbf{U}_t$ with eigenvalues $e^{\frac{i\omega t}{2}}$, $e^{-\frac{i\omega t}{2}}$, respectively.

$$\mathbf{U}_t|\psi_0\rangle = e^{\frac{i\omega t}{2}}cos(\frac{\theta}{2})|\uparrow\rangle + e^{-\frac{i\omega t}{2}}e^{i\phi}sin(\frac{\theta}{2})|\downarrow\rangle = e^{\frac{i\omega t}{2}}\left(cos(\frac{\theta}{2})|\uparrow\rangle + e^{i(\phi-\omega t)}sin(\frac{\theta}{2})|\downarrow\rangle\right)$$

The trajectory on Bloch sphere is shown below.



2) *[6 pts]*

$$
\begin{aligned}
E(t) = \langle \psi_t | \mathbf{H} | \psi_t \rangle &= -\frac{\hbar\omega}{2} \langle \psi_t | \sigma_z | \psi_t \rangle \\
&= -\frac{\hbar\omega}{2} \Big(cos(\frac{\theta}{2}) | \uparrow \rangle + e^{-i(\phi - \omega t)} sin(\frac{\theta}{2}) | \downarrow \rangle \Big) \Big(cos(\frac{\theta}{2}) \langle \uparrow | - e^{i(\phi - \omega t)} sin(\frac{\theta}{2}) \langle \downarrow | \Big) \\
&= -\frac{\hbar\omega}{2} (cos(\frac{\theta}{2})^2 - sin(\frac{\theta}{2})^2) = -\frac{\hbar\omega}{2} cos(\theta)
\end{aligned}
$$

where in the second equality we used the fact that $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenvectors of σ_z with eigenvalues 1, -1, respectively.
For $\theta = 0$, $E = -\frac{\hbar\omega}{2}$, which is the minimal energy.
For $\theta = \pi/2$, $E = 0$.
For $\theta = \pi$, $E = +\frac{\hbar\omega}{2}$, which is the maximal energy.

For variance we have

$$\mathbf{H}^2 = (\frac{\hbar\omega}{2})^2 \sigma_z^2 = (\frac{\hbar\omega}{2})^2 \mathbf{I}$$

$$\text{Var} = \langle \psi_t | \mathbf{H}^2 | \psi_t \rangle - \langle \psi_t | \mathbf{H} | \psi_t \rangle^2 = (\frac{\hbar\omega}{2})^2 - (\frac{\hbar\omega}{2})^2 cos(\theta)^2 = (\frac{\hbar\omega}{2})^2 (1 - cos(\theta)^2)$$

Variance vanishes for $\theta = 0, \pi$.

3) *[6 pts]*

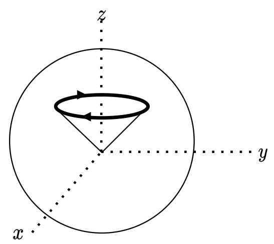
$$\rho_0 = \frac{1}{2}(\mathbf{I} + a_x \sigma_x + a_z \sigma_z) = \frac{1}{2} \begin{bmatrix} 1 + a_z & a_x \\ a_x & 1 - a_z \end{bmatrix}$$

$$
\begin{aligned}
\rho_t &= \mathbf{U}_t \rho_0 \mathbf{U}_t^\dagger \\
&= \begin{bmatrix} e^{\frac{i\omega t}{2}} & 0 \\ 0 & e^{-\frac{i\omega t}{2}} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 + a_z & a_x \\ a_x & 1 - a_z \end{bmatrix} \begin{bmatrix} e^{-\frac{i\omega t}{2}} & 0 \\ 0 & e^{\frac{i\omega t}{2}} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} e^{\frac{i\omega t}{2}} & 0 \\ 0 & e^{-\frac{i\omega t}{2}} \end{bmatrix} \begin{bmatrix} e^{-\frac{i\omega t}{2}}(1 + a_z) & e^{\frac{i\omega t}{2}} a_x \\ e^{-\frac{i\omega t}{2}} a_x & e^{\frac{i\omega t}{2}}(1 - a_z) \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 + a_z & e^{i\omega t} a_x \\ e^{-i\omega t} a_x & 1 - a_z \end{bmatrix}
\end{aligned}
$$

4) *[6 pts]* Note that we can write $\rho_t = \frac{1}{2}(\mathbf{I} + \overrightarrow{a_t}.\overrightarrow{\sigma})$, where

$$a_x(t) = a_x cos(\omega t), \quad a_y(t) = -a_x sin(\omega t), \quad a_z(t) = a_z$$

For $a = (1/2, 0, 1/2)$, the trajectory on bloch sphere is given below



5) *[6 pts]*

$$
\begin{aligned}
E(t) &= Tr[\mathbf{H}\rho_t] \\
&= -\frac{\hbar\omega}{4} Tr[\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1+a_z & e^{i\omega t}a_x \\ e^{-i\omega t}a_x & 1-a_z \end{bmatrix}] \\
&= -\frac{\hbar\omega}{4} Tr[\begin{bmatrix} 1+a_z & e^{i\omega t}a_x \\ -e^{-i\omega t}a_x & -1+a_z \end{bmatrix}] \\
&= -\frac{\hbar\omega}{2} a_z
\end{aligned}
$$

We see that the energy is independent of t and a_x.

For variance, we have

$$
\begin{aligned}
Var &= Tr[\mathbf{H}^2\rho_t] - Tr[\mathbf{H}\rho_t]^2 \\
&= (\frac{\hbar\omega}{2})^2 Tr[\mathbf{I}\rho_t] - (\frac{\hbar\omega}{2})^2 a_z^2 \\
&= (\frac{\hbar\omega}{2})^2 Tr[\frac{1}{2} \begin{bmatrix} 1+a_z & e^{i\omega t}a_x \\ e^{-i\omega t}a_x & 1-a_z \end{bmatrix}] - (\frac{(\hbar\omega}{2})^2 a_z^2 \\
&= (\frac{\hbar\omega}{2})^2 (1-a_z^2)
\end{aligned}
$$

Variance vanishes for $a_z = \pm 1$, which are the states $|\uparrow\rangle, |\downarrow\rangle$.