Solutions to Problem Set 12

Exercise 1 - Control of plasma burn

(a) *Does a burning ITER plasma risk undergoing thermal instability?*

Thermal stability can be determined from the energy balance equation:

$$\frac{d(3\,nT)}{dt} = \frac{P_{in}}{V} + \frac{1}{4}n^2\left\langle\sigma v\right\rangle E_\alpha - \frac{3\,nT}{\tau_E(n,T)}. \tag{1}$$

As ignition is approached, it is justifiable to neglect P_{in} (i.e. set $P_{in} = 0$) because the α particles become the primary source of heating. Keeping in mind that we have assumed $n = const$, we then have

$$3n\frac{dT}{dt} \approx \frac{1}{4}n^2\left\langle\sigma v\right\rangle E_\alpha - \frac{3nT}{\tau_E(T)} \tag{2}$$

or

$$\frac{dT}{dt} = \frac{1}{12}n\left\langle\sigma v\right\rangle E_\alpha - \frac{T}{\tau_E(T)} \tag{3}$$

In equilibrium ($\frac{dT}{dt} = 0$), the solution is

$$\frac{nE_\alpha\tau_E}{12} = \frac{T_0}{\left\langle\sigma v\right\rangle} \tag{4}$$

where T_0 is the equilibrium temperature.

Consider now a small variation of temperature ΔT such that $T = T_0 + \Delta T$:

$$\begin{aligned}\frac{d\Delta T}{dt} &= \frac{1}{12}nE_\alpha\frac{d\left\langle\sigma v\right\rangle}{dT}\Delta T - \frac{\Delta T}{\tau_E} - T\frac{d\tau_E^{-1}}{dT}\Delta T & (5)\\ &= \frac{1}{12}nE_\alpha\frac{d\left\langle\sigma v\right\rangle}{dT}\Delta T - \frac{\Delta T}{\tau_E} + \frac{T}{\tau_E^2}\frac{d\tau_E}{dT}\Delta T & (6)\\ &= \frac{1}{\tau_E}\left[\frac{1}{12}nE_\alpha\tau_E\frac{d\left\langle\sigma v\right\rangle}{dT} - 1 + \frac{T}{\tau_E}\frac{d\tau_E}{dT}\right]\Delta T. & (7)\end{aligned}$$

If this variation occurs near equilibrium, we can substitute the equilibrium solution to obtain the final expression:

$$\frac{d\Delta T}{dt} = \frac{1}{\tau_E}\left[\frac{T}{\left\langle\sigma v\right\rangle}\frac{d\left\langle\sigma v\right\rangle}{dT} - 1 + \frac{T}{\tau_E}\frac{d\tau_E}{dT}\right]\Delta T \tag{8}$$

The stability of the solution is thus determined by the sign of $\left[\frac{T}{\langle\sigma v\rangle}\frac{d\langle\sigma v\rangle}{dT} - 1 + \frac{T}{\tau_E}\frac{d\tau_E}{dT}\right]$ (negative = stable, positive = unstable). The stability criterion hence becomes

$$\frac{T}{\tau_E}\frac{d\tau_E}{dT} < 1 - \frac{T}{\langle\sigma v\rangle}\frac{d\langle\sigma v\rangle}{dT} \tag{9}$$

We now use the scaling law for τ_E, considering only factors of T,

$$\tau_E \sim \frac{1}{T}(T^{0.5})^{-0.7}(T)^{-0.9}(T^{-2})^{-0.01} = T^{-1-0.35-0.9+0.02} = T^{-2.23}, \tag{10}$$

to estimate

$$\frac{T}{\tau_E}\frac{d\tau_E}{dT} \approx \frac{T}{\alpha T^{-2.3}}\alpha(-2.3\,T^{-3.3}) = -2.3. \tag{11}$$

Similarly,

$$\frac{T}{\langle\sigma v\rangle}\frac{d\langle\sigma v\rangle}{dT} \approx \frac{T}{AT^2}\frac{d(AT^2)}{dT} = 2. \tag{12}$$

This implies that the criterion

$$-2.3 < 1 - 2 = -1 \tag{13}$$

is satisfied, indicating that the plasma burn should be stable.

(b) *What measure can you think of to control the burn and prevent instability?*

- Decrease heating power (if any).
- Gas and/or impurity injection.
- Reduce magnetic field/turn off the coils.

Exercise 2 - On the use of Tritium in ITER and fusion reactors

(a) Calculation of the Tritium burn-up fraction for a generic fusion reactor:

$$f_B = \frac{1}{1 + \frac{1}{\tau_T\, n_T\, <\sigma v>_{DT}}} \tag{14}$$

In this expression, τ_T is the confinement time, n_T is the Tritium ion density, and $<\sigma v>_{DT}$ is the fusion reactivity for Deuterium-Tritium reactions. The burn-up fraction represents the fraction of Tritium that undergoes fusion reactions before being lost from the plasma. For the given parameters:

$$f_B = \frac{1}{1 + \frac{1}{2[\mathrm{s}]\cdot 2\times10^{20}[\mathrm{m}^{-3}]\cdot 2.475\times10^{-22}[\mathrm{m}^3/\mathrm{s}]}} \approx 9\% \tag{15}$$

Calculation of the Tritium burn-up fraction for ITER:

$$f_B = \frac{1}{1 + \frac{1}{\tau_T\, n_T\, <\sigma v>_{DT}}} \tag{16}$$

For ITER, the values of τ_T, n_T, and $< \sigma v >_{DT}$ are specific to its operating conditions:

$$f_B = \frac{1}{1 + \frac{1}{1[\mathrm{s}] \cdot 1\times10^{20}[\mathrm{m}^{-3}] \cdot 1.1\times10^{-22}[\mathrm{m}^3/\mathrm{s}]}} \approx 1.1\% \tag{17}$$

This lower burn-up fraction compared to a generic reactor reflects the specific design and operational parameters of ITER.

(b) Estimation of the Tritium mass burn rate for a generic reactor, considering thermal to electrical power conversion efficiency:

$$\frac{dM}{dt} \approx 56 \times \text{fusion power}(\mathrm{GW}_{\text{thermal}})\ [\mathrm{kg/year}] \tag{18}$$

The coefficient 56 converts the fusion power output into the annual mass burn rate of Tritium, accounting for the energy release per fusion reaction. Assuming the reactor produces 1 GW of thermal power and with a conversion efficiency of 40%:

$$\frac{dM}{dt} = 56 \times 1/0.4 = 140\ [\mathrm{kg/year}] \tag{19}$$

For ITER, which operates at 10% of the power output of a full-scale reactor:

$$\frac{dM}{dt} = 56 \times 0.1 \times 1/0.4 = 14\ [\mathrm{kg/year}] \tag{20}$$

(c) Determination of the Tritium inventory mass required for steady operation of a reactor:

$$M_0 \approx \frac{t_p \frac{dM}{dt}}{\eta_f f_B} \tag{21}$$

Where t_p is the plant operation time per year, η_f is the fueling efficiency, and f_B is the burn-up fraction. This expression estimates the necessary Tritium inventory to maintain continuous reactor operation. For the reactor:

$$M_0 = \frac{1 \times 140}{365 \times 0.5 \times 0.09} \approx 8.5\ \mathrm{kg} \tag{22}$$

For ITER:

$$M_0 = \frac{1 \times 14}{365 \times 0.2 \times 0.011} \approx 17.4\ \mathrm{kg} \tag{23}$$

This calculation incorporates the operational parameters and burn-up fraction to determine the required Tritium inventory, highlighting the greater demand for Tritium in ITER due to its lower burn-up efficiency.