Homework 11
Quantum Information Processing

Exercise 1 *W states, reduced density matrices, and von Neumann entropy.*

The W_θ state is defined here as

$$|W_\theta\rangle = \frac{\cos\theta}{\sqrt{2}}|100\rangle + \frac{\cos\theta}{\sqrt{2}}|010\rangle + \sin\theta|001\rangle$$

This and similar ones play an important role in quantum communication protocols. In this exercise we look at a few of its properties in order to illustrate the concept of reduced density matrix, partial trace, and von Neumann entropy. In what follows we assume that Alice, Bob and Charlie each have a one-qubit share of the state.

a) Compute the reduced density matrices ρ_C and ρ_{AB}.

b) What is the dimension of these matrices, their eigenvectors and corresponding eigenvalues of each density matrix ? Check that your results are consistent with the Schmidt theorem.

c) Compute the von Neumann entropy associated to the A and BC systems.

d) Now we want to show that the AB system is "Bell-non-local" in the sense that ρ_{AB} violates the Bell inequality if θ is small enough.

- Compute the average value of the Bell operator $\mathcal{B} = A\otimes B + A'\otimes B - A\otimes B' + A'\otimes B'$. Give the expression in terms of general angles α, β, α' and β'.

  Show that if $(\sin\theta)^2 < \frac{\sqrt{2}-1}{\sqrt{2}+1}$ the Bell inequality is *certainly* violated. *Hint*: this is not the best possible condition and can be obtained by a two line simple calculation (without carrying out any detailed optimization of the average value of the Bell operator).

- Now imagine that a general bipartite system has density matrix that can be written as a convex combination of product density matrices, i.e., $\rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)}$. Such states are called "separable". This is the generalization of the notion of product states introduced for pure state vectors. Mixed states that are not separable are called "entangled".

  Prove that for separable states we have $|\mathrm{Tr}\rho_{AB}\mathcal{B}| \leq 2$.

- From the above questions what can you conclude about ρ_{AB} when $(\sin\theta)^2 < \frac{\sqrt{2}-1}{\sqrt{2}+1}$?

Remarks: For pure states (vector states) there are two situations: product states satisfy the CHSH inequality and entangled states (i.e. non-product vector states) violate the CHSH inequality. For density matrices the situation is richer. Separable states satisfy the CHSH inequality, but for non-separable states the inequality may be violated or not. When it is violated the state is said to be "Bell-non-local" which is a strong form of entanglement. Non-separable states that do not violate the CHSH inequality are still entangled but this is a weaker form of entanglement (for example you cannot use them for a simple Ekert-like protocol).

The standard W state has $\sin\theta = \frac{\cos\theta}{\sqrt{2}} = \frac{1}{\sqrt{3}}$ leads to a ρ_{AB} that can be shown to not violate the CHSH inequality but is non-separable (entangled).

Exercise 2 *Dynamics of 1-qubit density matrix*

In class we showed that the general form of a 1-qubit density matrix is

$$\rho = \frac{1}{2}(I + \vec{a}\cdot\vec{\sigma})$$

where $\vec{a} = a_x, a_y, a_z)$ is a vector in the unit three dimensional ball (the Bloch *ball*) $\|\vec{a}\| \leq 1$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the three usual Pauli matrices. Consider the dynamics of this mixed state generated by the Hamiltonian of the qubit in a static plus rotating magnetic field in the rotating frame (as seen in class, $\omega_1 \propto$ the strength of the rotating field and and $\delta = \omega - \omega_0$ the detuning between the Larmor and rotating field frequencies)

$$H = \frac{\hbar\delta}{2}\sigma_z - \frac{\hbar\omega_1}{2}\sigma_x$$

a) Show that the density matrix at time t is of the form

$$\rho_t = \frac{1}{2}(I + \vec{a}(t)\cdot\vec{\sigma})$$

and compute the vector $\vec{a}(t)$. *Hint*: From the definition of the density matrix you can infer that

$$\rho_t = U_t \rho U_t^\dagger$$

with U_t the evolution operator.

b) Check that $\|\vec{a}(t)\| = \|\vec{a}\|$. So the vector $\vec{a}(t)$ evolves on a sphere (inside the Bloch ball) of radius given by the initial vector.

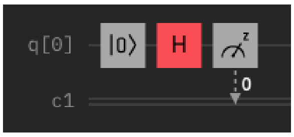
c) Find a simple proof of the last statement without ever computing $\vec{a}(t)$.

Exercise 3 *IBM Quantum Composer*

This exercise allows you to familiarize yourself with the IBM Q experience website, their graphical interface, the Qiskit language, simulators and NISQ devices. You will find many tutorials on IBM Q. It is important for you to familiarize yourself in order to solve the *mini-project* (graded) that we will distribute next week.

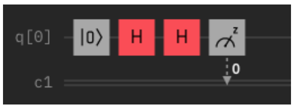
a) Create an IBM account and sign-in in IBM-Composer. You can find some tutorials and documentation at the following address.

b) Design your first circuit with a simple qubit initialized at $|0\rangle$ followed by a Hadamard gate and a measurement stored on one classical bit.

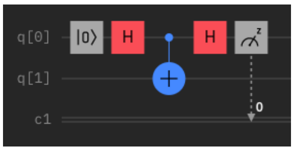


Click on the "Setup and run" button, and select a "system" device or a "simulator". A job will be sent to an IBM device, and the status of the computation can be checked on the left panel of the interface. What result do you get?

c) Now design a circuit with two Hadamard gates. Because $H^2 = I$, you know that the system should go back to its previous initial state $|0\rangle$ (as illustrated with the double beam-splitter interferometer at the beginning of the course). Check that this is the case.

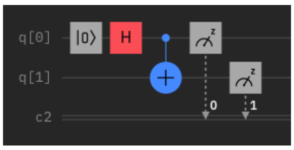


d) Add an interference with a second qubit with a control-not gate as follow:



From the point of view of the first qubit, this can also be understood as effectively entangling it with the "environment". Run the circuit, compare the results with the previous question. How does it affect the system?

e) Remove the last Hadamard gate and add a measurement on the second qubit. What do you obtain and why?



f) Now it is your turn to create a circuit to illustrate the w-state $|W\rangle$ seen in the course.