Homework 9
Quantum Information Processing

Exercise 1 *Bell states*

The purpose of this exercise is to get you familiar with calculations involving Bell states: the calculations of the first three questions are in Dirac notation.

1) Show that the four Bell states introduced in class form an orthonormal basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$.

2) Prove that the state $|B_{00}\rangle$ (or take any other Bell states you like) is entangled.

3) Let $|\gamma\rangle = \cos\gamma\,|0\rangle + \sin\gamma\,|1\rangle$ and $|\gamma_\perp\rangle = -\sin\gamma\,|0\rangle + \cos\gamma\,|1\rangle$. Show the identity (for all angles of polarization γ)

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}(|\gamma\rangle \otimes |\gamma\rangle + |\gamma_\perp\rangle) \otimes |\gamma_\perp\rangle\,.$$

4) Represent the four Bell states in coordinates. Use the canonical representation $|0\rangle = (1,0)^\top$ and $|1\rangle = (0,1)^\top$.

Exercise 2 *Entanglement by unitary operations*

Let $|x\rangle \otimes |y\rangle$ with $x, y = 0, 1$ be the four states of the "computational basis" (or canonical) for two qubits. We define the unitary operation CNOT (called "controlled-not")

$$\text{CNOT}\,|x\rangle \otimes |y\rangle = |x\rangle \otimes |x \oplus y\rangle \tag{1}$$

where $x \oplus y$ is the addition of bits modulo 2. This operation models certain magnetic interactions between the degree of freedom in spin and is responsible for the entanglement.

1) Verify that

$$|B_{xy}\rangle = (\text{CNOT})(H \otimes I)\,|x\rangle \otimes |y\rangle$$

using Dirac's notation. Is the part $(H \otimes I)\,|x\rangle \otimes |y\rangle$ already entangled? Justify your answer! (Note that the identity above is the reason for the index notation $|B_{xy}\rangle$).

2) Write down CNOT and $H \otimes I$ and their product in matrix notation. Check that the matrices are unitary.

3) From the unitarity of the matrices, give a compact proof of orthonormality for the states $|B_{xy}\rangle$.

4) Use the entanglement of $|B_{xy}\rangle$ to show there cannot exist two 2×2 unitary matrices U_A and U_B such that $\text{CNOT} = U_A \otimes U_B$ (in other words it is the control-not operation that entangles the two qubits in the identity above).

Exercise 3 *Tsirelson's bound*

In class we defined the observables

$$A = (+1)|\alpha\rangle\langle\alpha| + (-1)|\alpha_\perp\rangle\langle\alpha_\perp|, \qquad A' = (+1)|\alpha'\rangle\langle\alpha'| + (-1)|\alpha'_\perp\rangle\langle\alpha'_\perp|$$

in Alice's lab and

$$B = (+1)|\beta\rangle\langle\beta| + (-1)|\beta_\perp\rangle\langle\beta_\perp|, \qquad B' = (+1)|\beta'\rangle\langle\beta'| + (-1)|\beta'_\perp\rangle\langle\beta'_\perp|$$

in Bob's lab. Furthermore the correlation coefficient in the Bell experiment is (according to QM) $\langle\Psi|\mathcal{B}|\Psi\rangle$ where $|\Psi\rangle$ is the state shared by Alice and Bob, where the observable $\mathcal{B}$ is

$$\mathcal{B} = A \otimes B + A' \otimes B - A \otimes B' + A' \otimes B'$$

Remark: here we do not pre-suppose that $|\Psi\rangle$ is a Bell state. In this exercise we guide you through the proof of the *Tsirelson bound* which states that for any state $|\Psi\rangle \in \mathcal{C}^2 \otimes \mathcal{C}^2$ we have

$$\langle\Psi|\mathcal{B}|\Psi\rangle \leq 2\sqrt{2}$$

The proof is most quickly done using the notion of operator or matrix norm. However here we avoid using any such notion and manage with only the Cauchy-Schwarz and triangle inequalities.

1) Show the identity

$$\mathcal{B}^2 = 4I_A \otimes I_B - [A, A'] \otimes [B, B']$$

where we have defined the *commutators* between two matrices as $[M, N] = MN - NM$. *Hint:* check and use that $A^2 = A'^2 = I_A$ and $B^2 = B'^2 = I_B$.

2) Deduce that for any state $|\Psi\rangle \in \mathcal{C}^2 \otimes \mathcal{C}^2$ we have:

$$\langle\Psi|\mathcal{B}^2|\Psi\rangle \leq 8$$

Hints: remark that $[A, A'] \otimes [B, B'] = ([A, A'] \otimes I_B)(I_A \otimes [B, B'])$ and use the Cauchy-Schwarz inequality to estimate $|\langle\Psi|[A, A'] \otimes [B, B']|\Psi\rangle$. You then have to estimate terms like $\|[A, A'] \otimes I_B|\Psi\rangle\|$. You will estimate this using the triangle inequality and showing that this is less than 2.

3) Finally show that

$$\langle\Psi|\mathcal{B}|\Psi\rangle^2 \leq \langle\Psi|\mathcal{B}^2|\Psi\rangle$$

to deduce Tsirelson's bound.