Nuclear Fusion and Plasma Physics **EPFL**
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Solutions to Problem Set 8

Exercise 1 - Alfvèn waves

a) Consider a transverse wave in a string with tension S and mass per unit length M (Fig. 1).

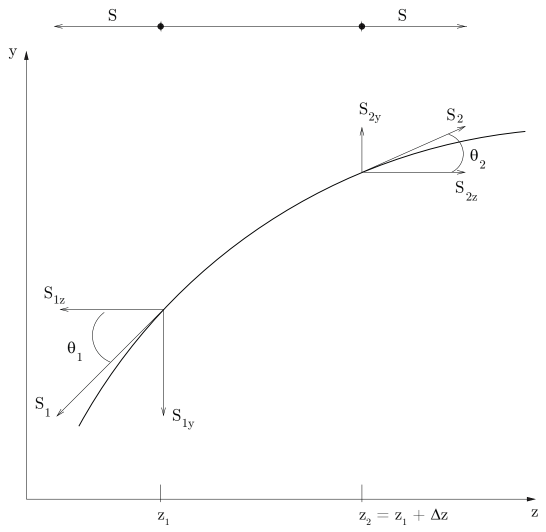

Figure 1: Transverse wave in a string.

For a purely transverse wave the net force along the z direction vanishes, so the net force is equal to the net force along y, F_y^{tot}.

$$|S_{1z}| = |S_{2z}| = S$$

Considering the geometry of the problem, we have the following relations:

$$\frac{S_{1y}}{S_{1z}} = \frac{S_{1y}}{S} = \tan\theta_1 = -\left(\frac{\partial y}{\partial z}\right)\Big|_z$$

$$\frac{S_{2y}}{S} = \tan\theta_2 = \left(\frac{\partial y}{\partial z}\right)\Big|_{z+\Delta z}$$

Using Newton's law in the y direction, we have

$$F_y^{tot} = m\frac{\partial^2 y}{\partial t^2} \Rightarrow S\left[\left(\frac{\partial y}{\partial z}\right)\Big|_{z+\Delta z} - \left(\frac{\partial y}{\partial z}\right)\Big|_z\right] = M\Delta z\,\frac{\partial^2 y}{\partial t^2}$$

where we used $m = M\Delta z$. Therefore,

$$S\frac{\left(\frac{\partial y}{\partial z}\right)\big|_{z+\Delta z} - \left(\frac{\partial y}{\partial z}\right)\big|_z}{\Delta z} = M\,\frac{\partial^2 y}{\partial t^2}$$

and, considering the limit $\Delta z \to 0$

$$S\frac{\partial^2 y}{\partial z^2} = M\frac{\partial^2 y}{\partial t^2} \Rightarrow \frac{\partial^2 y}{\partial z^2} = \frac{M}{S}\frac{\partial^2 y}{\partial t^2}.$$

This is the equation of a wave with velocity v given by:

$$v = \sqrt{\frac{S}{M}}$$

b) We are considering small perturbations on a uniform equilibrium:

$$\mathbf{u}_0 = 0;\ \ \rho_0,\ p_0 \text{ uniform};\ \ \mathbf{B}_0 = B_0\mathbf{e}_z$$

The linearized ideal MHD equations with respect to that equilibrium are:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}_1 = 0 \quad (1)$$

$$\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} = -\nabla p_1 + \frac{1}{\mu_0}(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 \quad (2)$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0) \quad (3)$$

$$p_1 = c_s^2 \rho_1 \quad (4)$$

We use the same geometry that has been considered during the course, with a magnetic field along z, $\mathbf{B} = B_0\,\hat{e}_z$ and the velocity perturbation in the y direction:

$$\mathbf{B_0} = (0, 0, B_0)$$
$$\mathbf{u} = (0, u_{1y}, 0)$$

Note: Following the analogy of the string, the magnetic field sets the direction of propagation of the wave (along z). As we have assumed a uniform equilibrium we can then set $\partial/\partial x = \partial/\partial y = 0$ since quantities do not change in the x and y directions.

Now, since $\partial/\partial y = 0$, we have $\nabla \cdot \mathbf{u} = 0$ (incompressible fluid) so that $\nabla \cdot \mathbf{u}_1 = 0$. Therefore $\rho_1 = 0$ and $p_1 = c_s^2 \rho_1 = 0$. Rewriting the cross products:

$$\mathbf{u}_1 \times \mathbf{B_0} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ 0 & u_{1y} & 0 \\ 0 & 0 & B_0 \end{vmatrix} = u_{1y} B_0 \hat{e}_x$$

$$\nabla \times (\mathbf{u}_1 \times \mathbf{B_0}) = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ u_{1y}B_0 & 0 & 0 \end{vmatrix} = B_0 \frac{\partial u_{1y}}{\partial z} \hat{e}_y$$

$$(\nabla \times \mathbf{B}_1) \times \mathbf{B_0} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ -\frac{\partial B_{1y}}{\partial z} & \frac{\partial B_{1x}}{\partial z} & 0 \\ 0 & 0 & B_0 \end{vmatrix} = B_0 \frac{\partial B_{1x}}{\partial z} \hat{e}_x + \ \ B_0 \frac{\partial B_{1y}}{\partial z} \hat{e}_y$$

The system of equations is then:

$$\rho_0 \frac{\partial u_{1y}}{\partial t} = \frac{B_0}{\mu_0} \frac{\partial B_{1y}}{\partial z} \tag{5}$$

$$\frac{\partial B_{1y}}{\partial t} = B_0 \frac{\partial u_{1y}}{\partial z} \tag{6}$$

If we consider the time derivative of Eq. (6),

$$\frac{\partial^2 B_{1y}}{\partial t^2} = B_0 \frac{\partial^2 u_{1y}}{\partial z \partial t};$$

we can substitute the term $\partial u_{1y}/\partial t$ from the Eq. (5) we find:

$$\frac{\partial^2 B_{1y}}{\partial t^2} = B_0 \frac{\partial}{\partial z} \left(\frac{B_0}{\mu_0 \rho_0} \frac{\partial B_{1y}}{\partial z} \right) = \frac{B_0^2}{\mu_0 \rho_0} \frac{\partial^2 B_{1y}}{\partial z^2}$$

$$\Rightarrow \quad \frac{\partial^2 B_{1y}}{\partial t^2} = \frac{B_0^2}{\mu_0 \rho_0} \frac{\partial^2 B_{1y}}{\partial z^2}$$

This is the equation of a wave propagating at the Alfvén velocity

$$c_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}}$$

and is formally equivalent to the equation for a transvere wave on a string with tension S and mass per unit length M. By comparing the propagation speeds

$$v = \sqrt{\frac{S}{M}} \Leftrightarrow c_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}}$$

one can then relate B_0/μ_0 to the tension and ρ_0/B_0 to the mass per unit length.

c) If we consider a plasma in ITER:

$$c_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}} \quad \text{with} \quad \rho_0 = m_p(A_D n_D + n_T A_T) = m_p n_e (A_D/2 + A_T/2)$$

$$c_A = \frac{6}{\sqrt{4\pi \cdot 10^{-7} \cdot 1.67 \times 10^{-27} \cdot 10^{20} \cdot (1 + 1.5)}} \simeq 8.27 \times 10^6 \text{ m s}^{-1}$$

d) In order to determine which particles are resonant with the Alfvén waves, we need to estimate the speed of the charged particles: Electrons, D ions, T ions and α particles. In order to do this, we use the formula of the thermal speed $v_{th,i} = \sqrt{\frac{T_i}{m_i}}$, where m_i and T_i indicated the mass and the (kinetic) energy of particle i respectively.

As the plasma is at $T = 13$ keV, we obtain $v_{\text{th},D} \simeq 7.9 \times 10^5$ m/s $v_{\text{th},T} \simeq 6.4 \times 10^5$ m/s, and $v_{\text{th},e} \simeq 4.8 \times 10^7$ m/s.

The speed of the α particles is estimated from the kinetic energy: $v_\alpha = \sqrt{\frac{2E_\alpha}{4m_p}} \simeq 1.3 \times 10^7$ m/s. These results mean that only the α particles are resonant with the Alfvén waves. Alphas are anyway slowed down by collisions, as we have seen in previous lectures.

Exercise 2 - CMA diagram

a) The X-mode cutoffs and resonances in terms of X, and Y are

- Cyclotron resonances $Y = 1/n^2$
- UH resonance: $1 = X + Y \rightarrow Y = 1 - X$
- Cutoff: $(\frac{\omega^2-\omega_p^2}{\omega^2})^2 - \frac{\Omega_e^2}{\omega^2} = 0 \rightarrow (1-X)^2 = Y$

b) Consider the case $n = 1$ (first harmonic heating). Initially, the wave is outside the plasma so the density is zero $\rightarrow X = 0$. The field at the edge is lower than at the center so $\Omega_e < \Omega_{e0} \rightarrow Y < 1$. As the wave propagates from low field side to high field side, the magnetic field increases as $B \sim 1/r$. At the same time, the density increases (X increases). At the plasma center the density is highest and $B = B_0 \rightarrow \Omega_e = \Omega_{e0} \rightarrow Y = 1$. At the high field side the density is again zero ($X = 0$) and the field is higher than at the center, so $Y > 1$. For nth harmionic heating the picture is exactly the same but the values of Y are centered around $1/n^2$. The propagation of the wave for different harmonics is presented in Fig. 2.

For first harmonic X-mode heating a wave launched from the LFS ($B < B_0$ so it starts *below* the resonance in the CMA diagram) first encounters the cutoff. It will therefore be reflected. However for 2nd harmonic heating and above it is possible for the wave to encounter the resonance first, provided the density is not too high. If it were possible to launch from the High Field Side (HFS), X1 heating would be possible as well.

c) O-mode cutoffs and resonances in terms of X and Y:

- Cyclotron resonances $Y = 1/n^2$
- Cutoff: $X = 1$

The O mode has fewer restrictions in terms of cutoff, the only cutoff being the plasma frequency which depends only on the density and is in any case at higher density than the X mode cutoff.

d) Based on the magnetic fields the resonances are ITER:170 GHz, TCV:41 GHz. This rules out X2 heating on ITER because this above the reach of present gyrotron technology. Indeed, ITER will use $>$ 20 MW of EC heating in the first harmonic O mode (O1). 170 GHz gyrotron sources capable of continuously delivering 2 MW are being studied and developed at SPC-EPFL.

TCV on the other hand can in principle use X2 heating. In practice it uses both X2 (3 MW launched from the low field side) and X3 (1.5 MW launched from the top). The advantage of X3 is that higher densities can be reached than with X2.

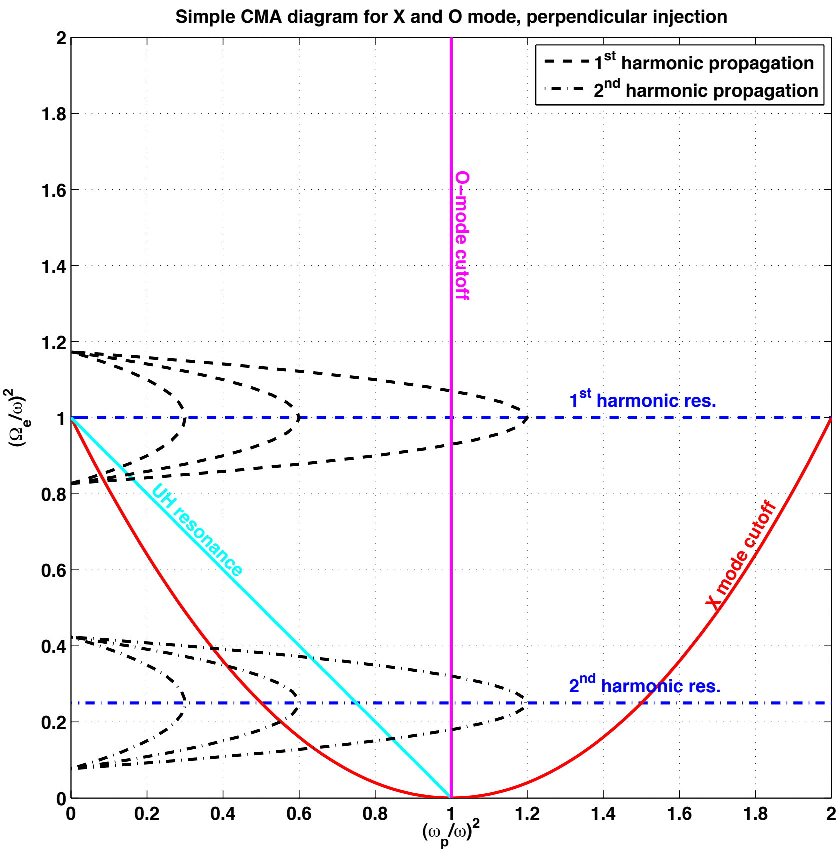

Figure 2: Clemmow-Mullaly-Allis diagram for X and O mode. Wave trajectories are shown for 1^{st} and 2^{nd} harmonic injection and for different core plasma densities. Note that for low field side X1 injection the wave first encounters a cutoff. X2 may encounter a cutoff or resonance, depending on the density. O mode has a higher density limit but will eventually be cut off at the plasma frequency.