Nuclear Fusion and Plasma Physics

Prof. A. Fasoli - Swiss Plasma Center / EPFL

Lecture 8 - 11 November 2024

General formalism for waves in the two fluid model

- Two-Fluid Model for $\boldsymbol{B}_0 \neq 0$, $T = 0$

Waves in the two-fluid model

- Cut-offs (all θ).
- Resonances for $\theta = \frac{\pi}{2}$
- Dispersion relation for $\theta = \frac{\pi}{2}$
- Use of dispersion relations.

The case of inhomogeneous plasmas

- Ray-tracing.
- Accessibility ("CMA" diagram).

Brief discussion of wave-particle interactions

- Wave-particle resonances.
- Collisionless (Landau) damping.
- Cyclotron resonances.

Appendix: Parallel propagation of waves in plasmas

Summary

We have seen that the dynamical response of a plasma to a perturbation (for a given equilibrium) evolves as a wave described in Fourier space by components (plane wave) behaving like

$$e^{\imath(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}.$$

As the system is linear (after linearisation, by considering only small perturbations to a given equilibrium), the final solution will be the sum (or the integral) of all the plane waves.

The key point in wave physics is to know which combinations of frequency and wavelengths $(\omega, \boldsymbol{k})$ can exist and propagate in the plasma.
$\omega = \omega(\boldsymbol{k})$, or $\boldsymbol{k} = \boldsymbol{k}(\omega)$ (both from the implicit relation $D(\omega, \boldsymbol{k}) = 0$) gives this information: “dispersion relation”.

The dispersion relation is the key to all wave physics problems and applications, from knowing which modes can become unstable in a burning plasma, to determining which sources and geometries to heat a plasma.

We have started with the most macroscopic model to describe the plasma, the MHD model. We have seen that in this model three kinds of waves can exist: the shear Alfvén $\omega^2 = k_z^2 c_A^2$ $(p_1 = \rho_1 = 0)$, the compressional Alfvén $\omega^2 = k^2 c_A^2$ and the sound wave (also compressional) $\omega^2 = k_z^2 c_S^2$.

For shear Alfvén waves, we have seen an analogy with a chord subject to tension and inertia. In the ideal MHD, the plasma is attached to field lines.

However, MHD is only a very crude description of the plasma: for example it does not account for distinction between species, hence it cannot describe resonance phenomena essential for plasma heating.

1 General formalism for waves in the two fluid model

- Infinite medium
- Small perturbations → linearisation + Fourier
- Idea: Maxwell's equations in vacuum, but with plasma as the source:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{1.1}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \cdot \mathbf{B} = 0 \tag{1.2}$$

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \frac{\partial \mathbf{j}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}; \quad \text{how to express } \mathbf{j} \text{ in terms of } \mathbf{E} \text{ ?} \tag{1.3}$$

In uniform and stationary conditions, the Fourier components are linked by a "simple" relation (constitutive relation of matter):

$$\mathbf{j}_{\omega,\mathrm{k}} = \underline{\underline{\sigma}}_{\omega,\mathrm{k}} \cdot \mathbf{E}_{\omega,\mathrm{k}} \qquad \underline{\underline{\sigma}}_{\omega,\mathrm{k}}\text{: conductivity tensor} \tag{1.4}$$

⇒ Fourier:

$$-\mathbf{k} \times (\mathbf{k} \times \mathbf{E}_{\omega,\mathrm{k}}) = i\omega\mu_0 \underline{\underline{\sigma}} \cdot \mathbf{E}_{\omega,\mathrm{k}} + \frac{\omega^2}{c^2}\mathbf{E}_{\omega,\mathrm{k}} \tag{1.5}$$

Multiplying by $\frac{c^2}{\omega^2}$ and noting that $c^2\mu_0 = \frac{1}{\varepsilon_0}$ one finds that:

$$-\frac{c^2}{\omega^2}\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = \left(\frac{i}{\omega\varepsilon_0}\underline{\underline{\sigma}} + \underline{\underline{1}}\right) \cdot \mathbf{E} \tag{1.6}$$

where $\underline{\underline{1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \delta_{ij}$ and $\left(\frac{i}{\omega\varepsilon_0}\underline{\underline{\sigma}} + \underline{\underline{1}}\right) = \underline{\underline{\epsilon}}$ dielectric tensor

As

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = k^2\left(\frac{\mathbf{kk}}{k^2} - \underline{\underline{1}}\right) \cdot \mathbf{E} \qquad \text{and} \qquad N^2 = \frac{k^2c^2}{\omega^2}, \tag{1.7}$$

the wave equation in Fourier space becomes

$$\left\{N^2\left(\frac{\mathbf{kk}}{k^2} - \underline{\underline{1}}\right) + \underline{\underline{\epsilon}}\right\} \cdot \mathbf{E} = 0 \tag{1.8}$$

Note 8.1.1: Plasma physics is in $\underline{\underline{\epsilon}}$, i.e. in $\underline{\underline{\sigma}}$. The crucial point is constructing the relation between $\underline{\underline{\sigma}}$ and $\mathbf{E}$.

Two-Fluid Model for $B_0 \neq 0$, $T = 0$

We will now consider plasma waves and oscillations with $\boldsymbol{B}_0 \neq 0$[1] in the cold plasma model, $T = 0$. We expect that $\boldsymbol{B}_0$, by introducing a "privileged" direction, will bring a wide variety of plasma modes of oscillation.

Let's take a two-fluid model with $T = 0$, and therefore $p = 0$, with an equilibrium

$$\boldsymbol{u}_{\alpha 0} = 0 \qquad\qquad \boldsymbol{B}_0 = B_0\boldsymbol{e}_z \tag{1.9}$$

where $\alpha = e, i$ denotes the plasma species and B_0, $n_{\alpha 0} \equiv n_0$ and ρ_0 are uniform. The linearisation of the equation of motion

$$m_\alpha\left\{\frac{\partial \boldsymbol{u}_\alpha}{\partial t} + (\boldsymbol{u}_\alpha \cdot \boldsymbol{\nabla})\boldsymbol{u}_\alpha\right\} = q_\alpha\left\{\boldsymbol{E} + \boldsymbol{u}_\alpha \times \boldsymbol{B}\right\} \tag{1.10}$$

[1] Most plasmas of interest, also because of flux freezing, have $\boldsymbol{B}_0 \neq 0$ somewhere

yields

$$m_\alpha \frac{\partial \boldsymbol{u}_{\alpha 1}}{\partial t} = q_\alpha \boldsymbol{E}_1 + q_\alpha \boldsymbol{u}_{\alpha 1} \times \boldsymbol{B}_0 \tag{1.11}$$

and after a Fourier transformation

$$-\imath \omega m_\alpha \boldsymbol{u}_{\alpha 1} = q_\alpha \boldsymbol{E}_1 + q_\alpha \boldsymbol{u}_{\alpha 1} \times \boldsymbol{B}_0. \tag{1.12}$$

Introducing the *mobility tensor* $\underline{\underline{\mu}}_\alpha$ this can be written as

$$\boldsymbol{u}_{\alpha 1} = \underline{\underline{\mu}}_\alpha \cdot \boldsymbol{E}_1. \tag{1.13}$$

Note that due to the $\boldsymbol{u}_\alpha \times \boldsymbol{B}_0$ term, $\underline{\underline{\mu}}_\alpha$ (hence $\underline{\underline{\sigma}}$ and $\underline{\underline{\varepsilon}}$) will not be diagonal. Careful separation of the components in eq.(1.12) yields

$$\underline{\underline{\mu}}_\alpha = \frac{q_\alpha}{m_\alpha} \begin{pmatrix} \frac{-\imath\omega}{\Omega_\alpha^2 - \omega^2} & \frac{\Omega_\alpha}{\Omega_\alpha^2 - \omega^2} & 0 \\ -\frac{\Omega_\alpha}{\Omega_\alpha^2 - \omega^2} & \frac{-\imath\omega}{\Omega_\alpha^2 - \omega^2} & 0 \\ 0 & 0 & \frac{\imath}{\omega} \end{pmatrix}. \tag{1.14}$$

The current density is given by

$$\boldsymbol{J} = \sum_\alpha q_\alpha n_{\alpha 0} \boldsymbol{u}_{\alpha 1} = \sum_\alpha q_\alpha n_{\alpha 0} \underline{\underline{\mu}}_\alpha \cdot \boldsymbol{E}_1 \equiv \underline{\underline{\sigma}} \cdot \boldsymbol{E}_1. \tag{1.15}$$

We get for the conductibility tensor

$$\underline{\underline{\sigma}} = \sum_\alpha q_\alpha n_{\alpha 0} \underline{\underline{\mu}}_\alpha = \sum_\alpha \frac{q_\alpha^2}{m_\alpha} n_{\alpha 0} \begin{pmatrix} \frac{-\imath\omega}{\Omega_\alpha^2 - \omega^2} & \frac{\Omega_\alpha}{\Omega_\alpha^2 - \omega^2} & 0 \\ -\frac{\Omega_\alpha}{\Omega_\alpha^2 - \omega^2} & \frac{-\imath\omega}{\Omega_\alpha^2 - \omega^2} & 0 \\ 0 & 0 & \frac{\imath}{\omega} \end{pmatrix}. \tag{1.16}$$

Finally we obtain the dielectric tensor

$$\underline{\underline{\varepsilon}} = \underline{\underline{\mathbb{1}}} + \frac{\imath \underline{\underline{\sigma}}}{\varepsilon_0 \omega} = \begin{pmatrix} \epsilon_1 & -\imath\epsilon_2 & 0 \\ \imath\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \tag{1.17}$$

where

$$\epsilon_1 = 1 + \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\Omega_\alpha^2 - \omega^2} \tag{1.18}$$

$$\epsilon_2 = -\sum_\alpha \frac{\Omega_\alpha}{\omega} \frac{\omega_{\mathrm{p}\alpha}^2}{\Omega_\alpha^2 - \omega^2} \tag{1.19}$$

$$\epsilon_3 = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2} \tag{1.20}$$

Note that, for a cold plasma, $\underline{\underline{\varepsilon}}$ does *not* depend on $\boldsymbol{k}$, but only on ω. For $\boldsymbol{B}_0 \to 0$ we have $\epsilon_2 \to 0$ and $\epsilon_1 \to \epsilon_3$, thus $\underline{\underline{\varepsilon}}$ becomes a diagonal matrix. As we have expected, there is no privileged direction anymore.

2 Waves in plasmas

2.1 Waves in the two fluid model

- Homogenous equation:

$$\det\left\{N^2\left(\frac{\boldsymbol{k}\boldsymbol{k}}{k^2}-\underline{\underline{\mathbb{1}}}\right)+\underline{\underline{\epsilon}}\right\}=0 \tag{2.1}$$

  to have non-trivial solution, i.e. $\boldsymbol{E}\neq 0$
- Choose a geometry: $\boldsymbol{B}_0=B_0\hat{z}$; $\boldsymbol{k}=(0,k\sin\theta,k\cos\theta)$

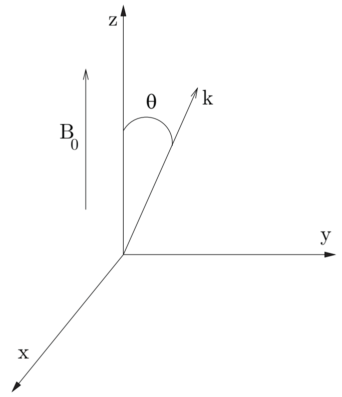


Figure 1: Notation: Geometry of magnetic field and wave.

Choosing $\boldsymbol{k}$ in the yz-plane and defining the angle θ with respect to the z-axis as shown in figure 1, we find

$$\begin{aligned}
N^2\left[\frac{\boldsymbol{k}\boldsymbol{k}}{k^2}-\underline{\underline{\mathbb{1}}}\right]+\underline{\underline{\epsilon}} &= \begin{pmatrix}-N^2 & 0 & 0\\ 0 & -N^2\cos^2\theta & N^2\sin\theta\cos\theta\\ 0 & N^2\sin\theta\cos\theta & -N^2\sin^2\theta\end{pmatrix}+\begin{pmatrix}\epsilon_1 & -\imath\epsilon_2 & 0\\ \imath\epsilon_2 & \epsilon_1 & 0\\ 0 & 0 & \epsilon_3\end{pmatrix}\\
&= \begin{pmatrix}-N^2+\epsilon_1 & -\imath\epsilon_2 & 0\\ \imath\epsilon_2 & -N^2\cos^2\theta+\epsilon_1 & N^2\sin\theta\cos\theta\\ 0 & N^2\sin\theta\cos\theta & -N^2\sin^2\theta+\epsilon_3\end{pmatrix}.
\end{aligned}$$

We impose the condition

$$\det\begin{pmatrix}-N^2+\epsilon_1 & -\imath\epsilon_2 & 0\\ \imath\epsilon_2 & -N^2\cos^2\theta+\epsilon_1 & N^2\sin\theta\cos\theta\\ 0 & N^2\sin\theta\cos\theta & -N^2\sin^2\theta+\epsilon_3\end{pmatrix}=0 \tag{2.2}$$

to have a non–trivial solution for $\boldsymbol{E}_1$. This leads to a dispersion relation of the type

$$AN^4+BN^2+C=0 \tag{2.3}$$

where A and B depend on the *angle* θ (between $\boldsymbol{k}$ and $\boldsymbol{B}_0$) and ω, but not on $|\boldsymbol{k}|$, and C only depends on ω.

Important points are

- **"cut-off"** where the wave is *reflected*

$$N = 0,\ C = 0 \qquad \Longrightarrow \qquad \frac{\omega}{k} \to \infty \qquad (k = 0,\ \omega \neq 0) \tag{2.4}$$

- **"resonance"** where the wave is *absorbed*

$$N \to \infty,\ A = 0 \qquad \Longrightarrow \qquad \frac{\omega}{k} \to 0 \tag{2.5}$$

> **Note 8.2.1**: To have a transfer of energy from the wave to the plasma (to heat it or to drive current), one has to inject a wave that avoids cut-off and reaches a resonance in the plasma.

Cut-offs

$N^2 \to 0 \Longleftrightarrow C \to 0$.

Introducing

$$\epsilon_R \equiv \epsilon_1 + \epsilon_2 \tag{2.6}$$

$$\epsilon_L \equiv \epsilon_1 - \epsilon_2 \tag{2.7}$$

we can write

$$C = \epsilon_R \epsilon_L \epsilon_3 \tag{2.8}$$

Note that C is independent of θ. In the cold plasma model, the cut–offs do not depend on the propagation angle. In general, there are three cut–offs

$$\epsilon_R = 0 \qquad \Longrightarrow \qquad \omega = \omega_R \tag{2.9}$$

$$\epsilon_L = 0 \qquad \Longrightarrow \qquad \omega = \omega_L \tag{2.10}$$

$$\epsilon_3 = 0 \qquad \Longrightarrow \qquad \omega \simeq \omega_{\rm pe} \tag{2.11}$$

In the limit $\Omega_e \gg \Omega_i$,

$$\omega_{R,L} \cong \frac{1}{2}\left\{\sqrt{\Omega_e^2 + 4\omega_{\rm pe}^2} \pm \Omega_e\right\} \quad , \tag{2.12}$$

thus $\omega_L \leq \omega_{\rm pe} \leq \omega_R$.

In the limit $\boldsymbol{B} \to 0$ we find that $\omega_{R,L} = \omega_{\rm pe}$.

These are points we need to avoid if we want to launch a wave in the plasma, for example to heat it.

Resonances

$N^2 \to \infty \Longleftrightarrow A \to 0$.

As the condition

$$A = A(\omega, \theta) = \epsilon_1 \sin^2\theta + \epsilon_3 \cos^2\theta = 0 \qquad (\text{if } \epsilon_1 \neq 0) \qquad (2.13)$$

depends on the angle θ, for given values of ϵ_1, ϵ_3 (*i.e.* of plasma parameters and frequency), there will be one angle for which the wave will encounter a resonance. Let's consider the perpendicular direction, $\theta = \pi/2$.

For $\theta = \frac{\pi}{2}$, to have $A \to 0$, we need.

$$\epsilon_1 \sin^2\theta + \epsilon_3 \cos^2\theta = \epsilon_1 = 0$$

This gives the so called "hybrid" resonances

$$\omega^2 \cong \Omega_e\Omega_i = \omega_{LH}^2 \qquad \text{"lower hybrid" resonance} \qquad (2.14)$$

$$\omega^2 \cong \omega_p^2 + \Omega_e^2 = \omega_{UH}^2 \qquad \text{"upper hybrid" resonance} \qquad (2.15)$$

Note 8.2.2: The lower hybrid resonance is very important for current drive in fusion.

Graphical summary of dispersion relation

Perpendicular propagation

$\theta = \frac{\pi}{2}$. We distinguish waves with $\boldsymbol{E} \parallel \boldsymbol{B}_0$ (so called Ordinary Mode, OM) and $\boldsymbol{E} \perp \boldsymbol{B}_0$ (so called Extraordinary Mode, XM).

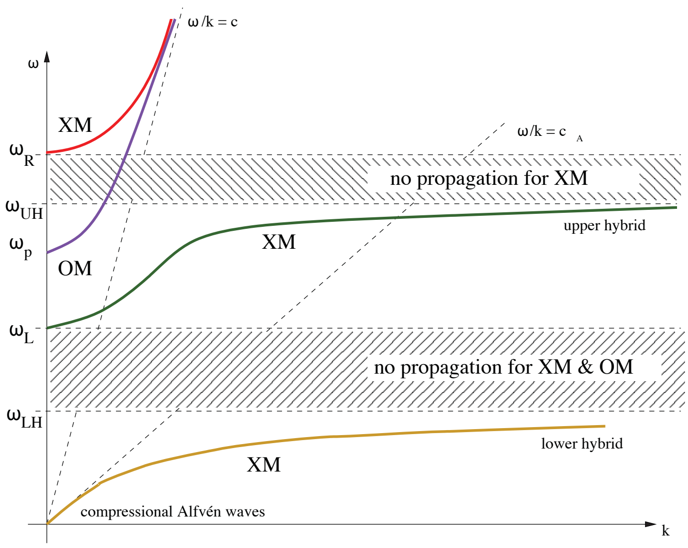


Figure 2: Dispersion relation for $\theta = \frac{\pi}{2}$

.

Note 8.2.3: The case of $\theta = \frac{\pi}{2}$ is particularly useful for heating fusion plasmas (access for antennas is typically from a side 'port').

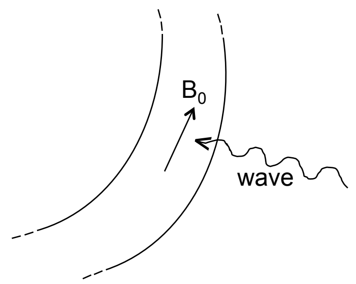


Figure 3: Access antennas by side port

2.2 Comments on the use of dispersion relations

Boundary value problem

In this type of problem the aim is to determine the expression of an electric field $\boldsymbol{E}(\boldsymbol{x}, t)$ of which we only know the value at one position in the plasma. We fix a boundary condition on the electric field, such as $\boldsymbol{E}(\boldsymbol{x} = 0, t)$ (e.g. with an antenna). The dispersion relation can be used in order to express the field $\boldsymbol{E}(\boldsymbol{x}, t)$ as an inverse Fourier transform from ω to t:

$$\boldsymbol{E}(\boldsymbol{x}, t) = \int_{\mathbb{R}} \mathrm{d}\omega \boldsymbol{E}_0(\omega) e^{i\left(\boldsymbol{k}(\omega)\cdot\boldsymbol{x} - \omega t\right)} \tag{2.16}$$

with $\omega \in \mathbb{R}$ and $\boldsymbol{k} \in \mathbb{C}^3$. The electric field evaluated in $\boldsymbol{x} = 0$ is

$$\boldsymbol{E}(0, t) = \int_{\mathbb{R}} \mathrm{d}\omega \boldsymbol{E}_0(\omega) e^{-i\omega t}.$$

Therefore the Fourier transform of $\boldsymbol{E}(0, t)$ with respect to t is:

$$\boldsymbol{E}_0(\omega) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}t \boldsymbol{E}(0, t) e^{i\omega t}$$

The electric field $\boldsymbol{E}(\boldsymbol{x}, t)$ can then be retrieved using Eq. 2.16. This solves the problem entirely, except that in several cases we don't have only a single root of the dispersion equation, but several. In these cases the boundary condition

$$\boldsymbol{E}(\boldsymbol{x} = 0, t)$$

alone is insufficient, and we also need as many derivatives as there are missing pieces of information. The electric field is now a linear combination of the solutions obtained from each root of the dispersion relation, noted $\boldsymbol{k}_j$, with amplitude $\boldsymbol{E}_{0j}$:

$$\boldsymbol{E}(\boldsymbol{x}, t) = \sum_{j=1}^{N} \int_{\mathbb{R}} \mathrm{d}\omega \boldsymbol{E}_{0j}(\omega) e^{i\left(\boldsymbol{k}_j(\omega)\cdot\boldsymbol{x} - \omega t\right)}$$

and thus

$$\left.\frac{\partial^m \boldsymbol{E}}{\partial \boldsymbol{x}^m}\right|_{\boldsymbol{x}=0} = \int_{\mathbb{R}} \mathrm{d}\omega \left\{ \sum_{j=1}^{N} \boldsymbol{E}_{0j}(\omega)\left[i\boldsymbol{k}_j(\omega)\right]^m \right\} e^{-i\omega t} \qquad \text{with } m = 0, ..., N-1.$$

where $\frac{\partial^m \boldsymbol{E}}{\partial \boldsymbol{x}^m}\big|_{\boldsymbol{x}=0}$, $m = 0, ..., N-1$, are the known boundary conditions. We are left with a system of N equations to find N unknowns, the amplitudes $\boldsymbol{E}_{0j}$, which provide a solution for the electric field.

Initial value problem

For this problem, the aim is also to determine the expression of an electric field $\boldsymbol{E}(\boldsymbol{x}, t)$ starting from an initial value. The procedure is the same as the boundary value problem, except that we have $\boldsymbol{E}(\boldsymbol{x}, t = 0)$, and we need to use the relation $\omega = \omega(\boldsymbol{k})$. The Fourier transforms will then go between $\boldsymbol{k}$ and $\boldsymbol{x}$.

Case of non-homogenous plasmas

Fusion plasmas are generally very non-homogenous ($n_e = n_e(\boldsymbol{r}), \boldsymbol{B}_0 = \boldsymbol{B}_0(\boldsymbol{r}), T_e = T_e(\boldsymbol{r}), ...$). How can our model, based on Fourier formalism, and on $\boldsymbol{J} = \underline{\underline{\sigma}} \cdot \boldsymbol{E}$ (i.e. on stationarity an uniformity), still hold? Are all of these dispersion relations still applicable in a non-homogenous plasma?

The key point is the ratio between wavelength and the scale of the spatial variation (and of course, between the wave period and the characteristic time of changes in the plasma). If $\lambda \ll L$ (for ex. $L = L_n = \frac{n}{\nabla n}$), and $\omega_{wave} \gg \frac{1}{\tau_{charac.}}$, then our formalism is still valid.

We "just" need to account for the fact that the dispersion relations are a function of position: $D_x(\omega, \boldsymbol{k})$ thus $\boldsymbol{k} = \boldsymbol{k}(\omega, \boldsymbol{x})$. At each $\boldsymbol{x}$ the relation $\boldsymbol{k}(\omega)$ is slightly different because the plasma parameters change. We can replace

$$e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} \qquad \text{by} \qquad exp\left(i\left\{\int_0^{\boldsymbol{x}} \boldsymbol{k}(\omega, \boldsymbol{x}')\cdot \mathrm{d}\boldsymbol{x}' - \omega t\right\}\right) \tag{2.17}$$

This is method is called "ray-tracing".

Plasma accessibility

Naturally, we need to explore the "accessibility[2]" to heat the plasma. As stated above, we need to reach a resonance by avoiding cut-offs. This can be visualised in a diagram ("CMA" diagram, see Fig. 4), which takes into account the two main parameters varying radially, n and B (for perpendicular propagation). For this purpose we define two quantities, X and Y, proportional to n and B_0^2 respectively:

$$X = \frac{\omega_p^2}{\omega^2} \ (\propto n) \qquad Y = \frac{\Omega_e^2}{\omega^2} \ (\propto B_0^2)$$

[2] This point was treated in today's problem set.

Using these quantities, the conditions for cut-offs (Eqs. 2.11 and 2.12) or resonance (Eq. 2.15) can be expressed as:

$$\text{Cut-off} \quad \begin{cases} \text{O-Mode}, & X = 1 \\ \text{X-Mode}. & Y = (1-X)^2 \end{cases} \tag{2.18}$$

$$\text{Resonance} \quad \begin{cases} \omega = \omega_{UH}, & Y = 1 - X \\ \omega = l\Omega_e, & Y = \frac{1}{l^2}(1, 0.25, ...)^* \end{cases} \tag{2.19}$$

* **Note 8.2.1** : these cyclotron resonances for perpendicular propagation are not in the fluid model; they exist only in the kinetic model.

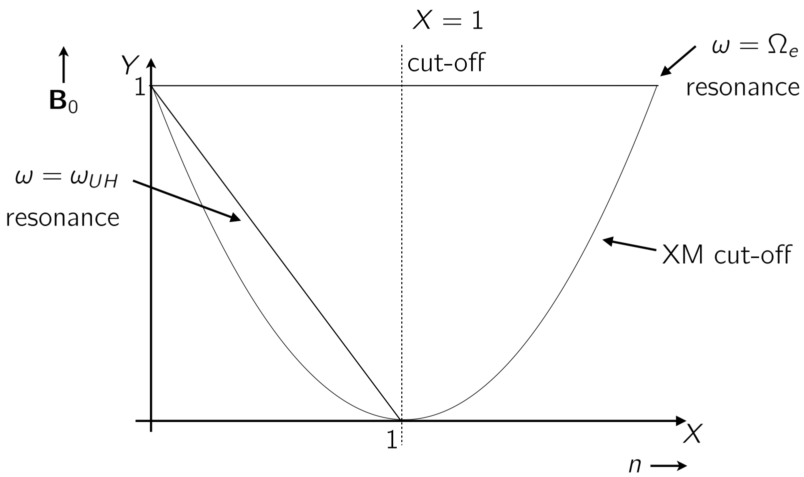


Figure 4: 'CMA' diagram illustrating the cut-offs and resonances

.

3 Kinetic Model

We have seen that the 'two-fluid' model leads to a variety of waves (in particular if $\boldsymbol{B}_0 \neq 0$), and to an idea of what happens to the waves in a real plasma.

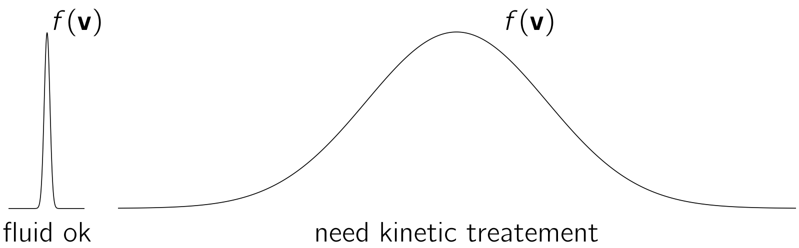
However, the fluid theory cannot describe the detail of the process of interactions between the waves and the plasma particles, which are important both for stability and for absorption (or damping) of the waves by the plasma.

For this, a 'kinetic' model is necessary, which describes the evolution of a distribution of particles, **not** all going at the same velocity.

Definition: The distribution function $f(\boldsymbol{x}, \boldsymbol{v}, t)$ is defined such that:

$f(\boldsymbol{x}, \boldsymbol{v}, t)\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{v}$ = the number of particles in $\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{v}$, phase space volume centered at $(\boldsymbol{x}, \boldsymbol{v})$, at time t.

The evolution of f is important when the velocities of the particles are quite different, i.e. for relatively large temperatures. Otherwise, when all particles have similar velocities, f is peaked and the fluid description is valid.

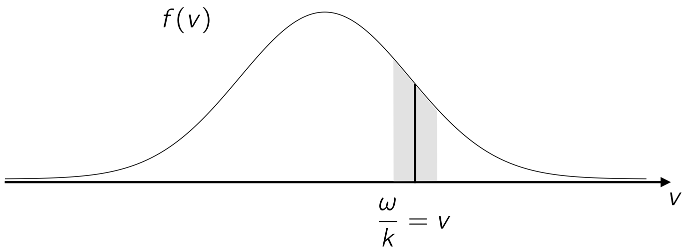


For high temperatures the plasma can be considered collisionless. As a reminder, this is because collision frequency scales like $T^{-3/2}$, so hotter plasmas are less collisional.

We will not study the details of the kinetic (also called 'hot plasma') model, but we will look (qualitatively) at one fundamental aspect of plasma waves.

3.1 Collisionless damping and wave-particle interaction

The key point in the energy exchange is the **wave-particle resonance**, which occurs when the particle moves roughly at the same velocity as the wave: $\mathbf{v}_{ph} = \frac{\omega}{k} \cong v_{particle}$.



The resonant particles are responsible for the exchange of energy with the wave.

The sign of particle acceleration depends on a phase term.

The question is if, overall, particles gain energy from the wave (damping, heating of plasma), or if the energy goes from the particles to the wave (instability). As $\omega = \omega_r + i\gamma$ and $E \propto e^{\imath\omega t}$, this is represented by the sign of γ, which we refer to as the "damping (or growth) rate". Indeed, the real part of ω goes into an oscillatory term so it is the sign of the imaginary part that is relevant: if $\gamma > 0$ then E decreases exponentially, and if $\gamma < 0$ it increases exponentially with time, resulting in an instability.

From the full theory, one finds that

$$\boxed{\gamma \propto \left.\frac{\mathrm{d}F_0}{\mathrm{d}u}\right|_{u=\frac{\omega}{k}}}, \tag{3.1}$$

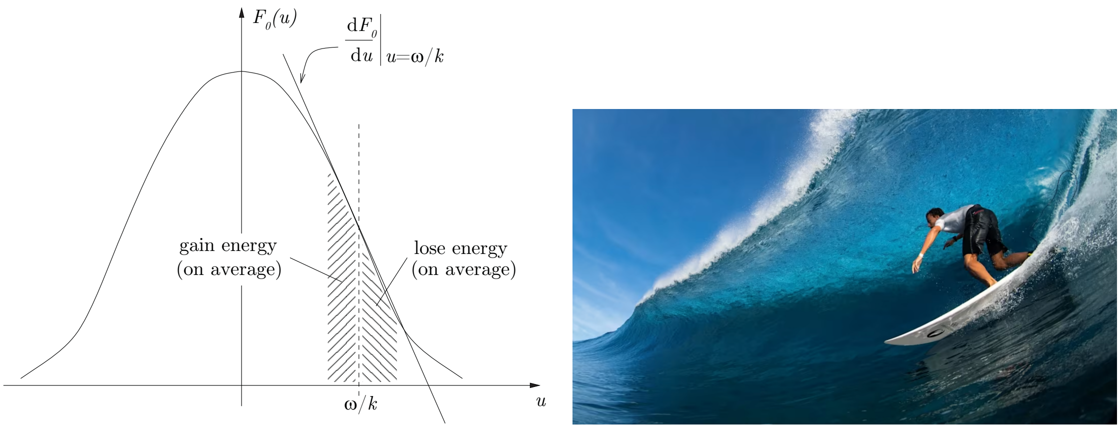
where F_0 is the unperturbed distribution function.

This is the **collisonless** or **Landau damping** (no need of collisions to exchange energy!)

Why is the damping rate proportional to the slope of F_0?

Consider particles with velocities just larger than the wave phase velocity $u \gtrsim \omega/k$. They can gain or lose energy depending on the relative phase of the wave, but if they gain energy, their velocity increases and they go out of the resonance: they can not exchange energy. If they lose energy, they slow down and stay longer in the resonance. So, overall, these particles *lose energy to the wave*.

The opposite holds for particles with velocities just below the phase velocity $u \lesssim \omega/k$. Those that gain energy from the wave remain in the resonance longer, and the net effect is that *particles gain energy from the wave*.



(a) (b)

Figure 5: (a) Particles with $u \lesssim \omega/k$ will gain energy from the wave and particles with $u \gtrsim \omega/k$ will lose energy to the wave. As there are more particles which gain energy, the overall effect is that the wave is damped.
(b) Analogy with a surfer riding a wave.

The total energy balance is therefore given by the ratio between how many particles gain energy from the wave (with $u \lesssim \omega/k$) and how many give energy to the wave ($u \gtrsim \omega/k$). This balance can be deduced from the slope of $F_0(u)$ around the resonance $u \simeq \omega/k$ (Fig. 5).

A (very) qualitative analogy can be drawn with surfers trying to catch an ocean wave: to 'ride' the wave (i.e. to be pushed by it) the surfer must prepare himself or herself more or less at the speed of the wave ($u \simeq \omega/k$), but just a little slower.

Question: if the wave is damped, its energy goes into the kinetic energy of the particles, but how can it happen without any collisions?

To understand this, we introduce the concept of phase mixing: microscopic (velocity dependent) perturbations of $f(v)$ around the resonance can remain (as there is no dissipation), but it is the collective motion of the particles that sustains a macroscopic perturbation. As a result, there can be a reduction in the wave amplitude due to the de-correlation of the individual velocity classes instead of dissipation:

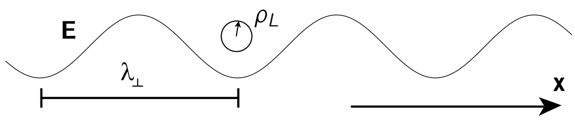
- if the initial perturbation of f, $f_{ini}(v) = \delta(v)$, there is no de-correlation, thus no damping
- if f_{ini} is wide there is de-correlation (phase mixing) and thus damping. The wider the distribution, the stronger the damping.

3.2 Cyclotron resonances

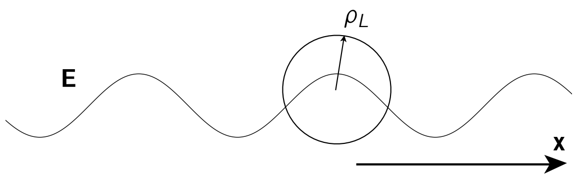
The collisionless absorption processes can be understood in terms of phase mixing and resonant wave-particle interaction. In fusion we are always in the presence of a magnetic field $\boldsymbol{B}_0$, so we have a special case of wave-particle resonance, at the cyclotron frequency (or its harmonics).

Consider a wave electric field perpendicular to $\boldsymbol{B}_0$ ($\boldsymbol{E} \perp \boldsymbol{B}_0$). In the presence of the magnetic field, a strong interaction between waves and plasma particles is only possible under certain conditions, at specific ratios between ω, the frequency of the wave electric field, and Ω, the frequency of gyromotion of the particles.

Fundamental frequency $\omega = \Omega$: a strong interaction is only possible if $\lambda_\perp \gg \rho_L$ (or $k_\perp \rho_L \ll 1$). Here, $\lambda_\perp$ is the wavelength of the wave electric field, and ρ_L (Larmor radius) is the radius of gyration of charges in the plasma due to $\boldsymbol{B}_0$.



In fact, if $\lambda_\perp \lesssim \rho_L$, we cannot guarantee that the particle motion remains in phase with the wave, which is a necessary condition for efficient exchange of energy.



First harmonic $\omega = 2\Omega$: a strong interaction is possible if $k_\perp \rho_L \sim 1$. If $\lambda_\perp \sim \rho_L$, the particle can encounter a field of the opposite sign in the second half of its gyromotion, so it can always be accelerated (or decelerated).

Higher harmonics $\omega = n\Omega$: to have resonance, the particle should have

$$v_{ph} \sim v_\perp = \Omega \rho_L \Rightarrow \frac{\omega}{k_\perp} \cong \Omega \rho_L \Rightarrow k_\perp \rho_L \simeq \frac{\omega}{\Omega} = n.$$

Note 8.3.1:

- A wave propagating exactly in a plane perpendicular to $\boldsymbol{B}_0$ cannot undergo cyclotron damping, and is not that useful for heating because only one velocity is resonant. However, if $k_\parallel \neq 0$, a finite portion of the distribution function can be resonant, i.e. absorb energy efficiently, as

$$\omega - k_\parallel v_\parallel = n\Omega$$

  where $\omega - k_\parallel v_\parallel$ is the Doppler shifted frequency.

- The same effect is produced by relativistic effects, as $\Omega \to \frac{\Omega}{\gamma}$ and for different energies, the resonant condition varies. Of course, for the relativistic effect to be significant, particles (electrons, in this case) need to be relatively energetic (high temperature).

Appendix: Parallel propagation of waves in plasmas

For $\theta = 0$ ($\boldsymbol{k} \parallel \boldsymbol{B}_0$), Eq. 2.13 becomes

$$\tan^2\theta = -\frac{\epsilon_3}{\epsilon_1} = 0 \tag{3.2}$$

Thus there are resonances for:

$$\epsilon_3 = 0 \quad \Longrightarrow \quad \omega^2 = \omega_{pe}^2 \quad \textit{see following note} \tag{3.3}$$

$$\epsilon_1 \to \infty \quad \Longrightarrow \quad \omega^2 = \Omega_{e,i}^2 \quad \text{“cyclotron resonances”} \tag{3.4}$$

> **Note**: The case $\epsilon_3 = 0, \omega^2 = \omega_{pe}^2$ is pathological: it is a cut-off and a resonance at the same time, which is unphysical. The problem is that we assumed $T = 0$; in reality for $T \neq 0$, it is only a cut-off.

Example of a full dispersion relation for parallel propagation

The idea is to split the electric field into two components with different polarisation (as in optics). Left and right polarizations are defined as:

$$E_R = E_x - iE_y \quad \to \text{rotates with the electrons (conter-clockwise)} \tag{3.5}$$

$$E_L = E_x + iE_y \quad \to \text{rotates with the ions.} \tag{3.6}$$

We therefore expect E_R and E_L to resonate with electrons and ions, respectively. The dispersion relation is given by:

$$N_{R,L}^2 = \frac{(\omega \mp \omega_R)(\omega \pm \omega_L)}{(\omega \pm \Omega_i)(\omega \mp |\Omega_e|)} \tag{3.7}$$

What is the limit of $N_{R,L}^2$ for $\omega, \boldsymbol{k} \to 0$?

$$\frac{k^2c^2}{\omega^2} \sim \frac{\omega_R\omega_L}{|\Omega_e|\Omega_i} = \frac{\omega_p^2}{|\Omega_e|\Omega_i} = \frac{e^2n}{\varepsilon_0 m_e}\frac{m_e}{eB_0}\frac{m_i}{eB_0} = \frac{m_i n}{\varepsilon_0 B_0^2} = c^2\frac{\rho_m}{B_0/\mu_0} = \frac{c^2}{c_A^2}$$

Thus, $\frac{k^2}{\omega^2} = \frac{1}{c_A^2}$, which corresponds to Alfvén waves. This is the MHD limit.

Idea to diagnose plasma: send a linear polarised wave, which can be seen as the sum of two circularly polarised waves, E_R and E_L.

The phase velocities of E_R and E_L are different (“bi-refringence”). Thus, rotation rates will be different. The vector $\boldsymbol{E}$ will rotate (depending on plasma parameters though ω_R, ω_L, Ω_e and Ω_i). The measure of the rotation of polarisation (also called Faraday rotation) allows one to measure B_0, the electron density n_e, etc. A schematic drawing is provided in Fig. 6.

Parallel propagation

For waves propagating parallel to $\boldsymbol{B}_0$ ($\theta = 0$), there are only the transverse wave branches, which exist only if $\boldsymbol{B}_0 \neq 0$. The graphical solution is displayed in Fig. 7.

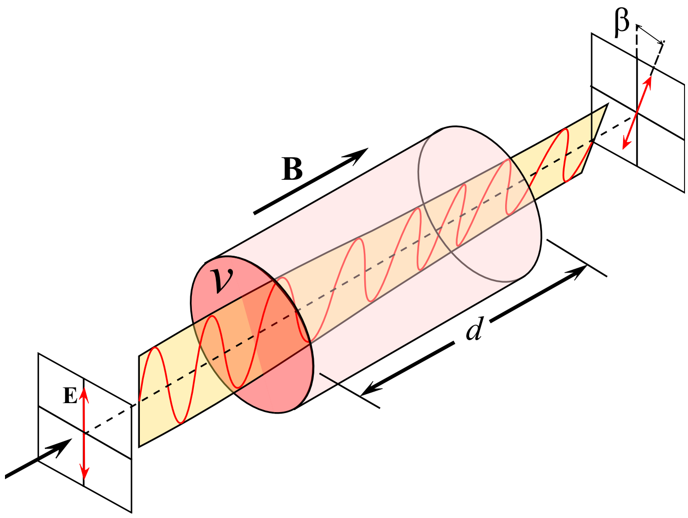


Figure 6: Faraday rotation: through the measure of β, describing the rotation of $\boldsymbol{E}$ in the plasma, it is possible to retrieve several parameters such as B_0 and n_e

.

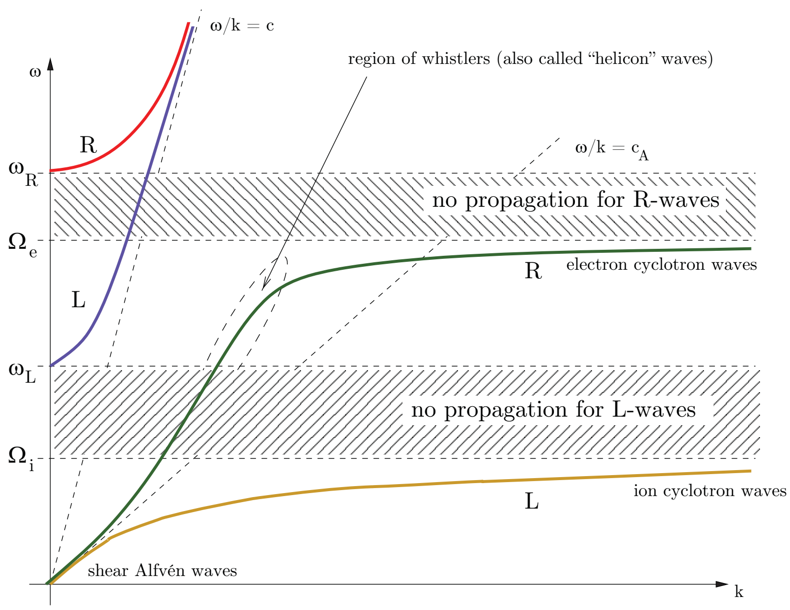


Figure 7: Graphical solution of the dispersion relation for $\theta = 0$ (parallel propagation).