Nuclear Fusion and Plasma Physics **EPFL**
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Solutions to Problem Set 6

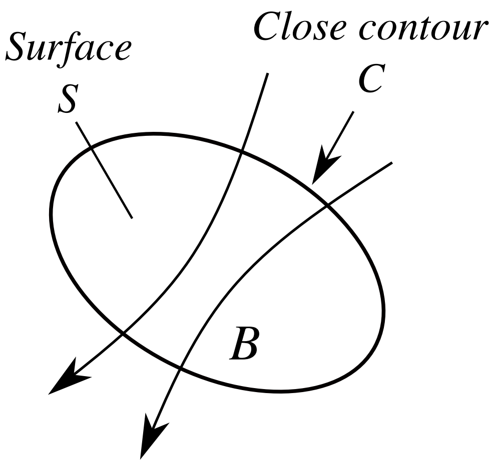
1 Exercise 1 - The frozen flux theorem

In what follows, please keep in mind that we use interchangeably the following notation for partial derivatives: $\partial_t \equiv \frac{\partial}{\partial t}$, $\partial_l \equiv \frac{\partial}{\partial l}$.

a) **Ideal MHD equations**

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1}$$
$$\rho \left[\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] = -\nabla p + \mathbf{j} \times \mathbf{B} \tag{2}$$
$$\frac{d}{dt}(p\rho_m^{-\gamma}) = 0 \qquad \text{State equation} \tag{3}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \tag{4}$$
$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \tag{5}$$
$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 \tag{6}$$

b) **Magnetic Flux through a surface S of closed contour C**



$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS} \tag{7}$$
$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{dS} \tag{8}$$
$$\frac{d\Phi}{dt} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS} + \int_S \mathbf{B} \cdot \frac{d}{dt}\mathbf{dS} \tag{9}$$

We could put the time derivative into the surface integral because the temporal and spatial coordinates are independent.

c) Equation (9) shows that the variation of the magnetic flux can happen either by a change of **B** inside the closed surface or a variation of the enclosing surface. We can rewrite each term of the equation separately.
Starting from the first one and using Faraday's law, we can write.

$$\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS} = -\int_S (\nabla \times \mathbf{E}) \cdot \mathbf{dS} \tag{10}$$

We use the Stokes' theorem to express the surface integral in an integral over the contour C, delimiting the surface.

$$\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS} = -\int_S (\nabla \times \mathbf{E}) \cdot \mathbf{dS} = -\oint \mathbf{E} \cdot \mathbf{dl} \tag{11}$$

The second term of equation (9) can be developed further by writing the time derivative of the surface element **dS** as a function of the fluid velocity **u**.

$$\mathbf{dS} = \mathbf{u}dt \times \mathbf{dl} \tag{12}$$

$$\int_S \mathbf{B} \cdot \frac{d}{dt}\mathbf{dS} = \oint \mathbf{B} \cdot (\mathbf{u} \times \mathbf{dl}) \tag{13}$$

Putting together the two contributions yields:

$$\frac{d\Phi}{dt} = -\oint_C \mathbf{E} \cdot \mathbf{dl} + \oint_C \mathbf{B} \cdot (\mathbf{u} \times \mathbf{dl}) \tag{14}$$

$$\frac{d\Phi}{dt} = -\oint_C (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \cdot \mathbf{dl} \tag{15}$$

Equation (15) has been obtained rearranging the terms in the second integral of (14) and combining the two integrals. From the ideal ohm's law, if follows that $\mathbf{E} + \mathbf{u} \times \mathbf{B} = \mathbf{0} \Rightarrow \frac{\mathbf{d\Phi}}{\mathbf{dt}} = \mathbf{0}$.

Exercise 2 - Cylindrical equilibrium

a) Given that

$$\mathbf{B} = B_0\hat{\mathbf{z}}, \quad p = p_0 \cos^2(\frac{\pi r}{2a}),$$

the MHD equilibrium is written as

$$\mathbf{J} \times \mathbf{B} = \nabla p$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Combining these expressions:

$$\nabla p = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{1}{\mu_0}\left[-\nabla\frac{B^2}{2} + (\mathbf{B}\cdot\nabla)\mathbf{B}\right]$$

$$\begin{aligned}\nabla\left[p + \frac{B^2}{2\mu_0}\right] &= \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} = \frac{\mathbf{B}}{\mu_0}(\hat{\mathbf{b}}\cdot\nabla)(\mathbf{B}\hat{\mathbf{b}}) \\ &= \frac{B^2}{\mu_0}(\hat{\mathbf{b}}\cdot\nabla)\hat{\mathbf{b}} + \hat{\mathbf{b}}(\hat{\mathbf{b}}\cdot\nabla)\frac{B^2}{2\mu_0}\end{aligned}$$

where we have used $\hat{\mathbf{b}} = \mathbf{B}/B$. The first term corresponds to the bending force of the B field. The second term corresponds to the compression of the field. In the case of straight and parallel field lines, the two terms on the right hand side are zero.
For the given profile of the magnetic field, this result can be easily retrieved by considering that:

$$\nabla p = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} = -\frac{1}{\mu_0}\frac{\partial B_z}{\partial r}\hat{\theta} \times B_z\hat{z} = -\frac{1}{\mu_0}\frac{dB_z}{dr}B_z\hat{r}$$

and as $\nabla p = \frac{dp}{dr}\hat{r}$, the above expression is equivalent to:

$$\nabla\left[p + \frac{B_z^2}{2\mu_0}\right] = 0$$

As a result,

$$p(r) + \frac{B_z^2}{2\mu_0} = p(a) + \frac{B_z^2(a)}{2\mu_0} = \frac{B_0^2}{2\mu_0} \quad (p(a) = 0)$$

So the maximum $p(r)$ would occur for $B_z(r) = 0$ and will have a value $p_{0,max} = \frac{B_0^2}{2\mu_0}$.

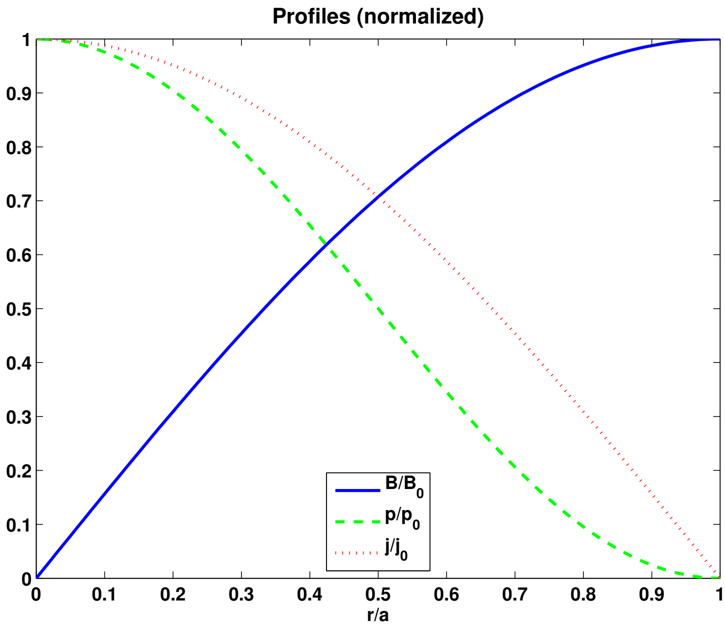
b) Starting from the relation between $p(r)$ and $B_z(r)$, we can determine the expression of the magnetic field:

$$\begin{aligned}B_z^2 &= B_0^2 - 2\mu_0 p_0 \cos^2(\frac{\pi r}{2a}) = B_0^2\left[1 - \cos^2(\frac{\pi r}{2a})\right] \\ \mathbf{B} &= B_0 \sin(\frac{\pi r}{2a})\hat{\mathbf{z}}\end{aligned}$$

The expression of the current is then easily determined using the MHD equilibrium relation between current and magnetic field:

$$\mathbf{J} = \frac{1}{\mu_0}\nabla \times \mathbf{B} = \frac{1}{\mu_0}\left(-\frac{\partial B_z}{\partial r}\right)\hat{\theta} = \frac{-\pi B_0}{2\mu_0 a}\cos\left(\frac{\pi r}{2a}\right)\hat{\theta}$$

c) Using the expressions derived in question b), the profiles of pressure, magnetic field and current are presented in the figure below as a function of the radius:


Profiles (normalized)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
r/a
B/B0
p/p0
j/j0