Nuclear Fusion and Plasma Physics

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Fluid description of plasmas

The MHD model: summary of its implications

- Flux freezing in ideal MHD and its consequences for plasma confinement and stability

MHD equilibrium

- Basic system of equations
- Magnetic tension and pressure terms
- Examples of equilibrium configurations
  - the z-pinch
  - the θ-pinch
  - the force-free equilibrium
  - bending the z-pinch into a torus

MHD stability

- General discussion on stability
- Example of instabilities: sausage and kink instability of a z-pinch.
- General interchange instability, good and bad curvature regions
- Methods to stabilise a plasma

1 MHD model

The MHD model is a single fluid description of a plasma, for macroscopic, large scale, relatively slow phenomena.

Ideal MHD In ideal MHD, the resistivity η is assumed to be zero, which implies the freezing of magnetic flux into the plasma (ex. dynamo effect, solar flares). In other words, the magnetic flux contained within any surface moving with the plasma is constant. Defining the magnetic flux as

$$\Phi(t) = \int_{S(t)} \mathbf{B} \cdot \mathrm{d}\mathbf{S}$$

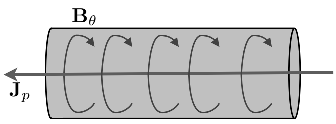
this condition is expressed as:

$$\frac{d\Phi}{dt} = 0$$

It shows that as the plasma moves, the magnetic flux is conserved, so we say that it is frozen into the plasma.

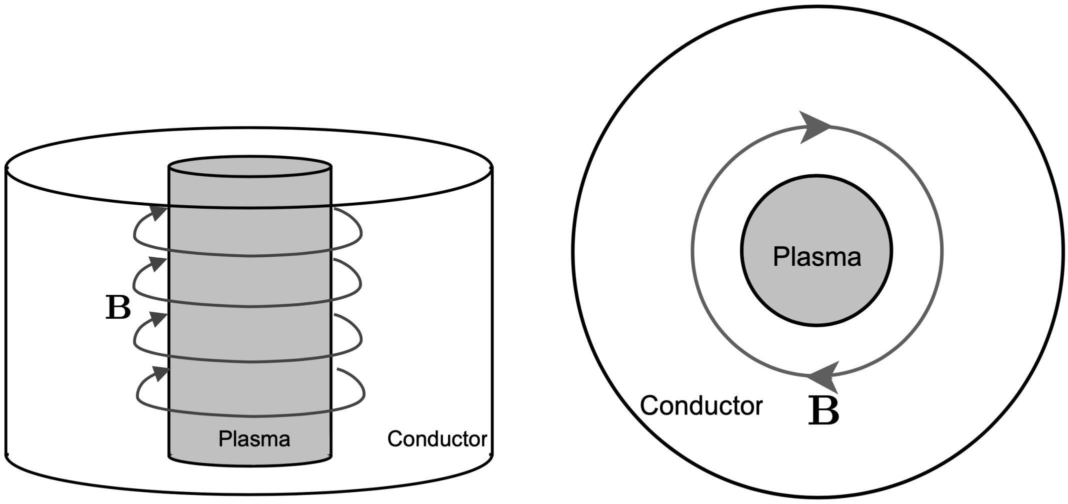
Application of flux freezing to fusion: plasmas can be shaped or stabilised by magnetic fields

Example 1: Linear z-pinch



- $\mathbf{J}_p$ (plasma current) produces $\mathbf{B}_\theta$.
- Increasing $\mathbf{J}_p$ increases $\mathbf{B}_\theta$ which compresses the plasma (flux must be conserved - in this case it is the flux on any azimuthal surface).

Example 2: B-field produced by external sources, plasma inside conductor

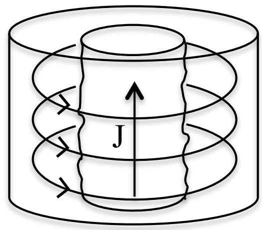


(a) (b)

Figure 1: View from the side (1a) and from above (1b) of a plasma confined in a magnetic field inside a conductor.

If **B** is increased, the plasma must be compressed as field lines cannot move through it. The conductor must be solid enough mechanically!

Example 3: Stabilisation of a plasma instability by a conducting wall



- Plasma has current densitiy **J**.
- An instability develops.
- If plasma moves toward the wall, flux conservation tends to keep the plasma in the center.

Resistive MHD In this case, as a result of collisions between plasma particles, the resistivity is no longer assumed to be zero. This implies that the **B**-field can diffuse with respect to the plasma. The equation governing the evolution of the **B**-field is derived starting from Maxwell's equations and Ohm's law:

$$\begin{cases} \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad , \quad \nabla \times \mathbf{E} = -\frac{\partial \mathrm{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \quad , \quad \mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{J} \end{cases} \tag{1.1}$$

Using the vector identity $\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$, we find that:

$$\frac{\partial \mathbf{B}}{\partial t} = \underbrace{\nabla \times (\mathbf{u} \times \mathbf{B})}_{\text{convection}} + \underbrace{\frac{\eta}{\mu_0} \nabla^2 \mathbf{B}}_{\text{diffusion}} . \tag{1.2}$$

The characteristic time for diffusion of the magnetic field is hence given by $\tau = \frac{\mu_0 L^2}{\eta}$, where L is the typical scale length. This can be very long (several seconds) for fusion plasmas, as η is small due to the high temperature, and L is generally large. Finite resistivity of the plasma (and of the conductor outside) limits the beneficial effects of flux freezing for confinement and stabilization to diffusion times.

1.1 Ideal MHD equilibrium

The static equilibrium condition on a fluid element is defined by $\partial/\partial t = 0$ and an equilibrium velocity $\mathbf{u} = 0$. Under these conditions, the ideal MHD system becomes:

$$\begin{cases} \mathbf{J} \times \mathbf{B} = \nabla p \quad , \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \\ \nabla \cdot \mathbf{B} = 0 \quad , \quad \nabla \cdot \mathbf{J} = 0 \end{cases} \tag{1.3}$$

Note 6.1.1:

- Notice that the first equation in Eq. 1.3 is simply the Force equation with $\mathbf{u} = 0$.
- By multiplying the force equation by $\mathbf{B}$, we see that:

$$\mathbf{B} \cdot (\boldsymbol{\nabla} p) = \mathbf{B} \cdot (\mathbf{J} \times \mathbf{B}) = 0 \qquad (\boldsymbol{\nabla} p \perp \mathbf{B}) \tag{1.4}$$

  Thus, pressure is constant along magnetic field lines: $\mathbf{B}$ lies on surfaces of constant pressure, also called **magnetic surfaces.**
  More generally, a magnetic surface $\Psi(\mathbf{r})$ is defined as a surface that is tangent to the magnetic field in every point, which implies that the scalar product between the gradient of the surface and $\mathbf{B}$ is null in every point:

$$\mathbf{B} \cdot \nabla \Psi = 0.$$

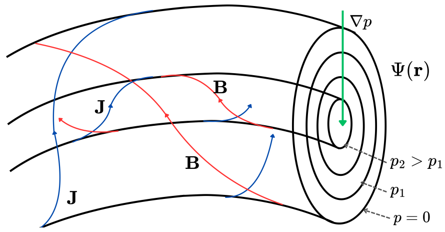
- By multiplying the force equation by $\mathbf{J}$, we see that:

$$\mathbf{J} \cdot (\boldsymbol{\nabla} p) = \mathbf{J} \cdot (\mathbf{J} \times \mathbf{B}) = 0 \tag{1.5}$$

  $\mathbf{J}$ also lies on surfaces of constant pressure.

  Thus, isobaric surfaces = magnetic surfaces = current surfaces (in ideal MHD equilibrium). Therefore, in ideal MHD equilibrium, isobaric surfaces are both magnetic surfaces and current surfaces (surfaces tangent to the current in every point.)

  Note, however, that although they lie on the same surfaces, $\mathbf{J}$ and $\mathbf{B}$ are not necessarily aligned (or orthogonal). This gives a simple way to represent the plasma equilibrium.



- Force balance is $\mathbf{J} \times \mathbf{B} = \boldsymbol{\nabla} p$, which can be expressed in a more intuitive way:

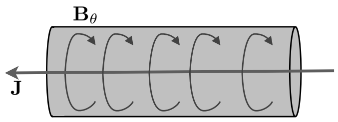
$$\boldsymbol{\nabla} p = \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} = \left(\underbrace{\frac{B^2}{\mu_0}(\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{b}}_{\text{tension}} - \underbrace{\boldsymbol{\nabla}_\perp \left(\frac{B^2}{2\mu_0} \right)}_{\text{pressure}} \right)$$

  where we have used $\mathbf{b} = \mathbf{B}/B$, $\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B}) = \boldsymbol{\nabla}\left(B^2/2\right) - (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B}$ and $\boldsymbol{\nabla}_\perp = \boldsymbol{\nabla} - \mathbf{b}(\mathbf{b} \cdot \boldsymbol{\nabla})$.

  Tension has the effect of a "restoring" force, acting when B-lines are bent. It scales with $\sim \frac{B^2}{\mu_0 R_c}$, R_c being the radius of curvature of the magnetic field lines.

Examples of MHD equilibrium configurations

Example 1: Linear z-pinch



This is a linear configuration where we consider a current flowing along the z-axis creating a magnetic field purely in the θ direction. Note that **B**, **J** and p are all solutions to the equilibrium equation.

To sum up, in cylindrical coordinates we have: $p = p(r)$, $\mathbf{B} = B_\theta(r)\hat{\theta}$, and $\mathbf{J} = J_z(r)\hat{z}$.

<u>Note 6.1.2</u>: The curl of a vector **A** in cylindrical coordinates is given by

$$\begin{cases} (\nabla \times \mathbf{A})_r = \dfrac{1}{r}\dfrac{\partial A_z}{\partial \theta} - \dfrac{\partial A_\theta}{\partial z} \\ (\nabla \times \mathbf{A})_\theta = \dfrac{\partial A_r}{\partial z} - \dfrac{\partial A_z}{\partial r} \\ (\nabla \times \mathbf{A})_z = \dfrac{1}{r}\dfrac{\partial r A_\theta}{\partial r} - \dfrac{1}{r}\dfrac{\partial A_r}{\partial \theta} \end{cases}$$

Plugging our **B** field in Ampere's law yields:

$$\frac{1}{\mu_0 r}\frac{\mathrm{d}}{\mathrm{d}r}(rB_\theta) = J_z$$

For a complete calculation we need to assume a current density profile. Let's take the simplest case $J_z = const = \frac{I_p}{\pi a^2}$, with I_p being the total plasma current and a the radius of the cylinder. Thus

$$\frac{1}{\mu_0 r}\frac{\mathrm{d}}{\mathrm{d}r}(rB_\theta) = \frac{I_p}{\pi a^2} \Rightarrow rB_\theta = \frac{\mu_0 r I_p}{\pi a^2}\frac{r}{2} + const \underset{B_\theta(0)=0}{\Rightarrow} B_\theta(r) = \frac{\mu_0 r}{2\pi a^2} I_p, \text{ for } r \leq a.$$

Note that for $r > a$, it's the usual application of Ampere's law:

$$B_\theta(r) = \frac{\mu_0 I_p}{2\pi r}, \text{ for } r > a$$

From force balance (Eq. 1.3):

$$\frac{dp}{dr} = -J_z B_\theta$$

Thus, for $r \leq a$ as this is valid inside the plasma:

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -J_z \frac{\mu_0 I_p}{2\pi a^2} r \Rightarrow p(r) = -\frac{\mu_0 I_p^2}{2\pi^2 a^4}\frac{r^2}{2} + const$$

The constant can be found from boundary conditions: as $p(a) = 0 \Rightarrow const = \frac{\mu_0 I_p^2}{2\pi^2 a^4}\frac{a^2}{2}$. Finally, the pressure (inside the plasma) is given by:

$$p(r) = \frac{\mu_0 I_p^2}{4\pi^2 a^2}\left[1 - \left(\frac{r}{a}\right)^2\right] \tag{1.6}$$

Fig. 2 illustrates the results derived for the case of the z-pinch discussed above.

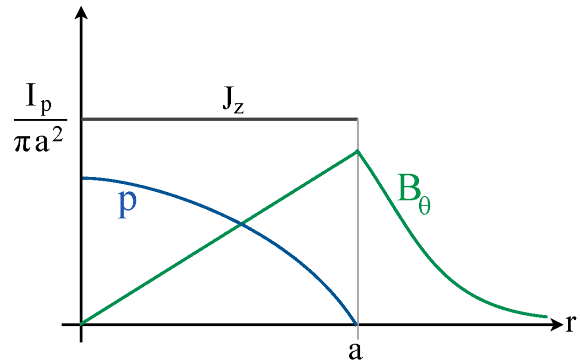


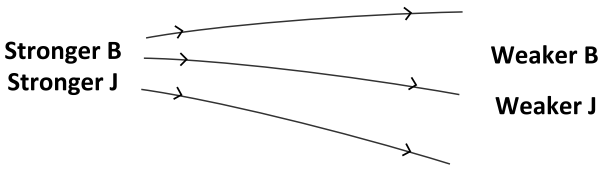
Figure 2: Magnetic field profile and pressure profile in a z-pinch for a given current density profile.

Example 2: Low-β plasma

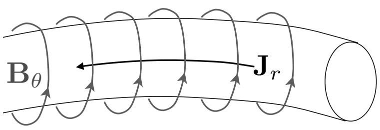
We define $\beta = \frac{p}{B^2/2\mu_0}$ as the ratio of the thermal to magnetic pressure and consider the case $\beta \ll 1$, meaning that the pressure term $p \sim 0$. So, from Eq. 1.3, $\mathbf{J} \times \mathbf{B} \cong 0$, implying that $\mathbf{J}$ is parallel to $\mathbf{B}$ or $\mathbf{J} = \mu(r)\mathbf{B}$ with μ a scalar quantity. As a result, current can only flow along $\mathbf{B}$, not across it. As $\nabla \cdot \mathbf{J} = 0$,

$$\nabla \cdot (\mu(r)\mathbf{B}) = \mu(r)\nabla \cdot \mathbf{B} + (\mathbf{B} \cdot \nabla)\mu(r) = 0 \Rightarrow (\mathbf{B} \cdot \nabla)\mu(r) = 0$$

This means that μ is constant along the magnetic field line, i.e. the ratio J/B is constant along the field line. A little like a "normal" fluid in "physical" conduits.

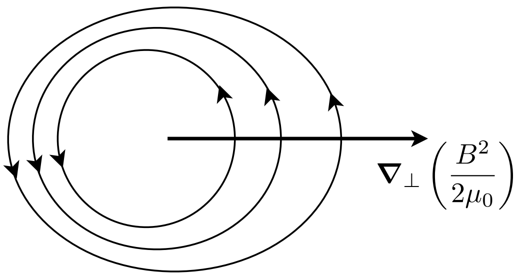


Example 3: Toroidal equilibrium



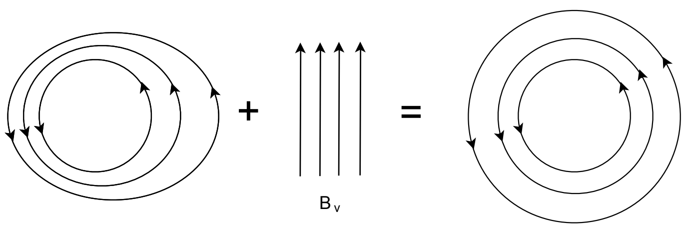
We now consider a configuration in which the z-pinch is bent into a torus.

The field B_θ is generated by the plasma current: hence, it is stronger inside than outside. On a cross-section:



The pressure force is outwards: known "hoop force". This force can be visualised by imagining many conductors repelling each other as they carry current in the opposite direction.

How can we reinforce the field on the outside, and weaken it to the inside? With a vertical field!



This vertical magnetic field produces a Lorentz force $\mathbf{J} \times \mathbf{B}_v$ that pushes back the plasma (of course, implying that we have a toroidal current too!). This is part of the tokamak concept (it needs an external vertical field for equilibrium), which will be discussed in the next lecture.

2 MHD stability

For any practical purpose, achieving an equilibrium is a necessary condition to have a confined plasma but is not necessarily sufficient to hold it for macroscopic time scales. For this we need to have a stable equilibrium.

Mechanical analogy

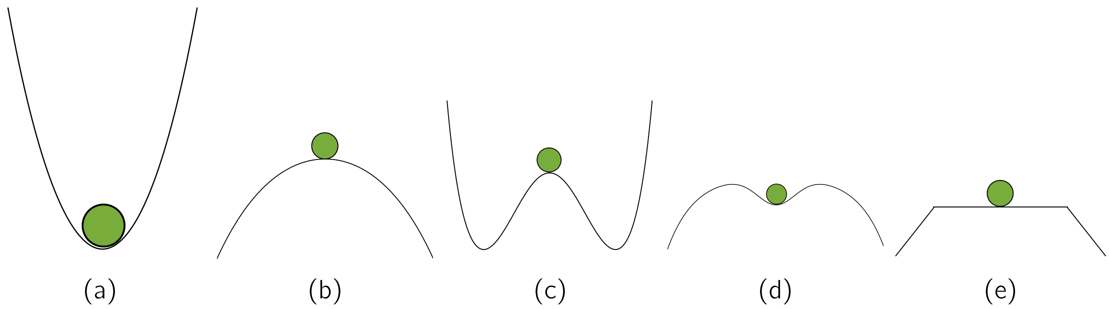


Figure 3: Stable (3a), unstable (3b), linearly unstable but non-linearly stable (3c), linearly stable but non-linearly unstable (3d), metastable (3e).

Perturbations that lower/increase the potential energy correspond to unstable/stable situations.

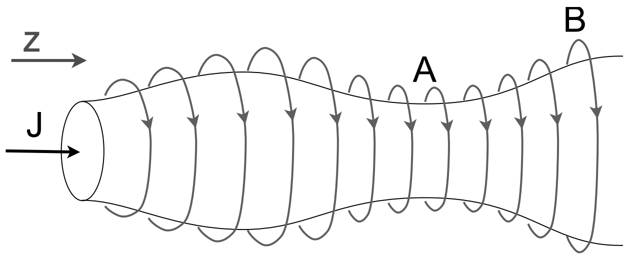
To study the stability in 3D (fusion relevant) situations, one needs to consider all possible perturbations to equilibrium. How can this be achieved?

- Macroscopic "interchange" of flux tubes (we know flux is frozen-in to the plasma): does energy increase or decrease when the flux tubes are "interchanged"? (stable vs. unstable, and how fast does the instability grow?)
- Fourier analysis of *small* perturbations (linearisation of MHD equations). Alternatively, normal mode analysis: consider periodic perturbations $\propto \exp(i\mathbf{k}\cdot\mathbf{x} - \imath\omega t)$ or, in a cylinder, $\propto \exp(ikx + im\theta - i\omega t)$, $m \in \mathbb{Z}$. This is a useful approach in uniform plasmas. $Im(\omega)$ will give us the growth rate (positive or negative) of the instability.
- In non-uniform plasmas, we take the MHD equation of motion and impose a small displacement $\boldsymbol{\xi}$ to fluid elements with respect to their equilibrium position. In a first order approximation, the velocity is hence given by $\mathbf{u} = \frac{\partial \xi}{\partial t}$. This can be substituted in the MHD force equation, yielding an expression of the form $\ddot{\boldsymbol{\xi}} = \mathbf{F}(\boldsymbol{\xi})$, where $\mathbf{F}(\boldsymbol{\xi})$ is the force operator. This is the Lagrangian point of view. Still, the issue is the sign of the change of the energy. For normal modes one gets an eigenvalue equation: $A\dot{\boldsymbol{\xi}} = \omega^2\dot{\boldsymbol{\xi}}$. As a result, the sign of ω^2 determines the stability, resulting either in an oscillating solution (stable) or an exponential increase of ξ in time (unstable).

Note 6.2.1: The methods of Fourier analysis and linearisation mentioned above will be discussed in more detail the next lecture.

Examples of instabilities

Example 1: z-pinch with "sausage perturbation"

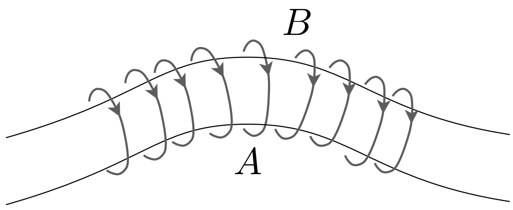


$\exp(\imath k_z z + \imath m\theta)$,
$k_z \neq 0, m = 0$

At point A, B_θ is stronger. This results in a tension $\frac{B_\theta^2}{\mu_0 r}$ and pressure $\frac{B_\theta^2}{\mu_0}$ increase, the perturbation increases as the inward force ∇p is not balanced.

At point B, B_θ is weaker and $\frac{B_\theta^2}{\mu_0 r}$ is smaller: the kinetic pressure is not balanced and pushes the plasma out which in turn causes the perturbation to increase. This therefore drives an **instability**.

Example 2: z-pinch with kink



$k_z \neq 0, m = 1$
B is stronger in A than in B: the perturbation grows, leading again to an **instability**

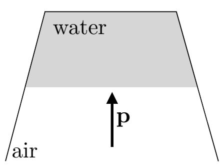
Note 6.2.2: The same configuration can be unstable with respect to different kinds of perturbations.

How would you act to stabilise the z-pinch? By adding B_z (z/θ-pinch): B_z provides "tension" along z.

In general, bending field lines requires energy, so the presence of **B**-tension along a given direction is stabilising. Perturbations leading to instabilities tend to "avoid" bending field lines.

2.1 Intercharge instability (Rayleigh-Taylor)

Example: ordinary fluids of different density

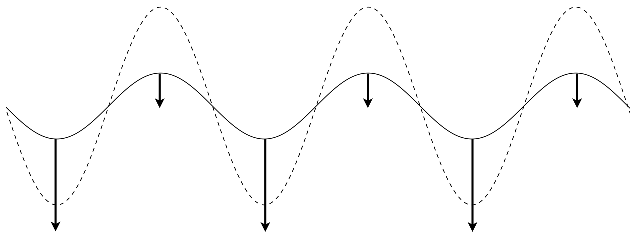


Consider a glass of water turned upside-down.

Air pressure would be enough to hold the water up, so the net force is zero ⇒ equilibium.

However (as we know from experience), this equilibrium is unstable. This is the case in general whenever a heavier fluid sits on top of a light fluid.

Any ripple/perturbation at the water/air interface will increase and drive the instability (**Rayleigh-Taylor instability**).



> **Note 6.2.3**: This is an example of interchange instability, as we can think of replacing the two volume elements at the fluid interface.

Plasma

Consider a situation similar to that seen above: plasma on top ("heavier" fluid), vacuum with magnetic field ("lighter" fluid) at the bottom, still in the presence of gravity.

Now introduce a small perturbation to the interface:

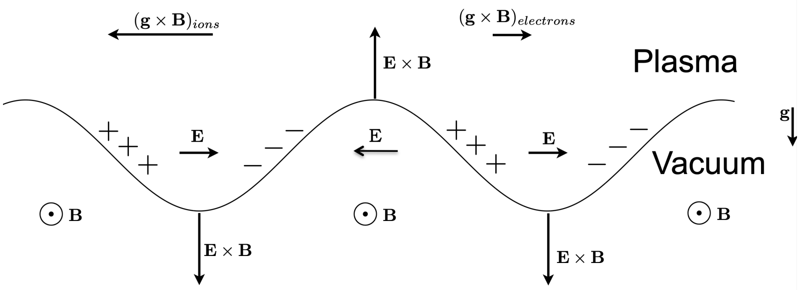


Figure 4: Effect of a small perturbation at the interface between plasma and vacuum.

Plasma particles are subject to a gravitational drift. The drift velocity is:

$$\mathbf{v}_g = \frac{m}{q}\frac{\mathbf{g} \times \mathbf{B}}{B^2} \qquad (2.1)$$

(much stronger for ions!) An electric field will be set-up, giving rise to $\mathbf{E} \times \mathbf{B}$, which will *increase* the perturbation.

One could argue that gravity is, in general, not important in plasma dynamics. In fact, in most cases of interest the effect of gravity in the mechanism of the Rayleigh-Taylor instability is replaced by that of B-field gradient or curvature.

Example: unstable vs. stable

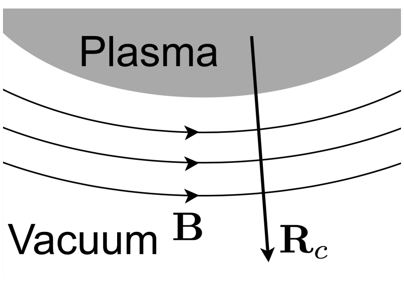


Figure 5: Bad curvature ⇔ unstable

The drift velocity caused by the centrifugal force is:

$$\mathbf{v}_{curv} = \frac{m}{q} v_{||}^2 \frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B^2} \qquad (2.2)$$

The same mechanism as above applies, if we replace the gravitational force by the centrifugal force:

$$m\mathbf{g} \longrightarrow \frac{m v_{||}^2 \mathbf{R}_c}{R_c^2}$$

As $\mathbf{R}_c$ points from the "heavier" fluid (plasma) to the "lighter" fluid (vacuum with **B**) we have **instability**. This is a case of "bad curvature": The curvature of the field points away from the region of higher pressure. The plasma is unstable when it is immersed in a magnetic field that is *concave* towards the plasma.

The opposite case is of course **stable**.

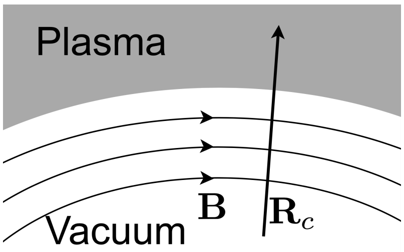


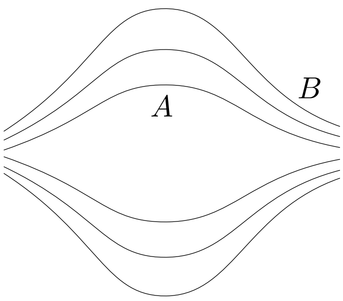
Figure 6: Good curvature ⇔ stable

This corresponds to having **g** pointing from the lighter fluid to the heavier fluid.

The Rayleigh-Taylor instability is a special case of a general class of instabilities, *interchange* instabilities, in plasma. These instabilities arise when by swapping the position of two flux tubes, the energy of the system decreases. They are of fundamental importance at the plasma-vacuum interface.

Note 6.2.4: The criteria for instability must be considered globally. In a real configuration there will be destabilising regions and stabilising regions. The balance between the two will determine the global stability.

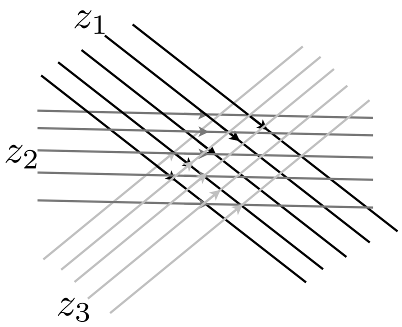
Example: Magnetic mirror



A is a bad curvature region
B is a good curvature region

The balance of the two regions may allow macroscopic stability, although there can be localised instabilities in the bad curvature region.

In addition to field line bending there is another mechanism that can help stabilising interchange instabilities: "*magnetic shear*"



At different depths in the plasma, B-field lines are directed along different directions: interchange becomes impossible unless we bend B-field lines, which has a **stabilising effect**.

This is the case of a tokamak: the “pitch” of the helical field structure depends on the radial position.