Nuclear Fusion and Plasma Physics

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October 26, 2026

Magnetic confinement

- The tokamak: concept and its main features
- The stellarator

Waves in plasmas

- Importance of plasma waves
- Mathematical techniques (linearisation, Fourier transform)
- Group and phase velocities
- Ideal MHD waves
  - shear Alfvén
  - compressional Alfvén and magnetosonic waves

1 Magnetic Confinement

Magnetic confinement is a method used in fusion reactors to confine hot plasma using magnetic fields. This method is essential in maintaining the plasma stability and achieving the necessary conditions for nuclear fusion.

1.1 The Tokamak

The tokamak is a type of magnetic confinement device that is axi-symmetric, meaning it has symmetry around its central axis. The tokamak design incorporates several key features:

- **Large Toroidal Magnetic Field**: This is the primary magnetic field in a tokamak, which is generated by the toroidal field (TF) coils. This field runs in the direction of the torus (doughnut-shaped) and is crucial for confining the plasma.
- **Small Poloidal Magnetic Field**: A secondary magnetic field that runs in small loops around the plasma. This field is essential for maintaining the plasma shape and stability.
- **Large Pressure**: The high pressure within the plasma is necessary to achieve the conditions for nuclear fusion.

The tokamak has four main components:

1. **Toroidal Field (TF) Coils**: These coils generate the large toroidal magnetic field that is the primary means of confining the plasma.
2. **Ohmic Heating (OH) Transformer**: This component induces a current in the plasma, providing heating and contributing to the equilibrium. The current also helps in maintaining the plasma's shape.
3. **Vertical Field System**: This system generates a vertical magnetic field, which is necessary for maintaining the toroidal force balance within the tokamak.
4. **Shaping Coils**: hese coils are used to improve MagnetoHydroDynamic (MHD) stability and alleviate interactions between the plasma and the reactor walls. Proper shaping of the plasma helps in achieving better confinement and reducing instabilities.

1.2 The Stellarator

The stellarator is another type of magnetic confinement device, which achieves the rotational transform from external coils only. Unlike the tokamak, the stellarator does not rely on a plasma current for confinement.

- **No need for plasma current**: The stellarator is designed to maintain plasma confinement purely through the external magnetic fields generated by its complex coil configurations, eliminating the need for a plasma current that is necessary for tokamaks.

- **Steady-state**: Since there is no reliance on a plasma current, which in tokamaks is typically pulsed, stellarators can operate in a steady-state manner, offering the potential for continuous operation without the disruptions associated with plasma current instabilities.

⇒ See viewgraphs for a qualitative discussion.

2 Waves in plasmas

All plasma particles are "sources" for Maxwell's equations. Therefore most dynamical processes in plasmas are related to electromagnetic waves and oscillations. Waves are used to heat plasmas, and to drive current non–inductively. Another example of the importance of waves is the role that microscopic electromagnetic waves and instabilities play in the transport of particles and energy in plasmas well above the levels due to collisional effects.

2.1 Mathematical technique

We will use normal mode (or plane wave) analysis. This corresponds to considering all quantities in Fourier space, using the Fourier transform defined for any quantity **g** as

$$\widetilde{\mathbf{g}}(\mathbf{k},\omega) = \frac{1}{(2\pi)^4}\int d^3x \int dt\, \mathbf{g}(\mathbf{x},t) e^{-i(\mathrm{k}\cdot\mathrm{x}-\omega t)}, \tag{2.1}$$

with the inverse transform given by

$$\mathbf{g}(\mathbf{x},t) = \int d^3k \int d\omega\, \widetilde{\mathbf{g}}(\mathbf{k},\omega) e^{i(\mathrm{k}\cdot\mathrm{x}-\omega t)}. \tag{2.2}$$

This will lead to complex quantities. Naturally, all physical quantities are real, and we will need to consider the real part at the end of all calculations.

The Fourier transformation is a linear operation. Its use comes from the fact that by using it we can split a complicated problem into small pieces, solve it for these small pieces, and combine the pieces to form the complete solution. This implies that the system of equations to be solved is linear.

When the system of equations to be solved is non-linear, we linearise it considering small perturbations to an existing equilibrium. Take for example the continuity equation (a differential equation) for the mass density ρ and the fluid velocity **u**:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{2.3}$$

where $\rho \equiv \rho(\mathbf{x},t)$ and $\mathbf{u} \equiv \mathbf{u}(\mathbf{x},t)$.

1. Choose an equilibrium → no time dependence → steady state:

$$\rho_0(\mathbf{x}) = \rho_0 \quad \text{(uniform equilibrium)}, \qquad \mathbf{u}_0(\mathbf{x}) = 0 \quad \text{(static equilibrium)} \tag{2.4}$$

2. Consider small perturbations to this equilibrium

$$\rho = \rho_0 + \rho_1(\mathbf{x}, t), \qquad \left|\frac{\rho_1}{\rho_0}\right| \ll 1 \quad \text{(expansion parameter)} \tag{2.5}$$

3. Linearise by retaining first order terms only to get the linearised continuity equation

$$\begin{aligned} \frac{\partial(\rho_0+\rho_1)}{\partial t} \quad &+ \quad \nabla\cdot\left((\rho_0+\rho_1)(\underbrace{\mathbf{u}_0}_{=0}+\mathbf{u}_1)\right) = 0 \\ \underbrace{\frac{\partial\rho_0}{\partial t}}_{\text{Order 0; = 0 by definition}} \quad &+ \quad \underbrace{\frac{\partial\rho_1}{\partial t}}_{\text{Order 1}} \quad + \quad \underbrace{\nabla\cdot(\rho_0\mathbf{u}_1)}_{\text{Order 1 and } \rho_0=\text{cte}} \quad + \quad \underbrace{\nabla\cdot(\rho_1\mathbf{u}_1)}_{\text{Order 2; neglected}} = 0 \\ \frac{\partial\rho_1}{\partial t} \quad &+ \quad \rho_0\nabla\cdot\mathbf{u}_1 = 0. \end{aligned} \tag{2.6}$$

4. Now we consider normal modes, i.e. we consider the perturbed quantities as Fourier transforms:

$$\rho_1(\mathbf{x}, t) = \int d^3k \int d\omega\, \widetilde{\rho}_1(\mathbf{k}, \omega) e^{i(\mathrm{k}\cdot\mathrm{x}-\omega t)} \tag{2.7}$$

and the same for $\mathbf{u}_1$. Thus

$$\begin{aligned} &\frac{\partial}{\partial t}\left\{\int d^3k \int d\omega\ \widetilde{\rho}_1(\mathbf{k},\omega)e^{i(\mathrm{k}\cdot\mathrm{x}-\omega t)}\right\} \\ + \quad &\rho_0\nabla\cdot\left\{\int d^3k \int d\omega\ \widetilde{\mathbf{u}}_1(\mathbf{k},\omega)e^{i(\mathrm{k}\cdot\mathrm{x}-\omega t)}\right\} = 0 \\ \Rightarrow \quad &\int d^3k \int d\omega\Big[-i\omega\widetilde{\rho}_1(\mathbf{k},\omega)\Big]e^{i(\mathrm{k}\cdot\mathrm{x}-\omega t)} \\ + \quad &\rho_0\int d^3k \int d\omega\left[i\mathbf{k}\cdot\widetilde{\mathbf{u}}_1(\mathbf{k},\omega)\right]e^{i(\mathrm{k}\cdot\mathrm{x}-\omega t)} = 0. \end{aligned} \tag{2.8}$$

In general we can make the following formal substitutions:

$$\nabla \to i\mathbf{k} \qquad \text{and} \qquad \frac{\partial}{\partial t} \to -i\omega. \tag{2.9}$$

In our example the linearised continuity equation becomes in Fourier space an algebraic equation:

$$-i\omega\widetilde{\rho}_1 + i\rho_0\mathbf{k}\cdot\widetilde{\mathbf{u}}_1 = 0. \tag{2.10}$$

In the following we will drop the tilde symbol to simplify the notation.

Note 7.2.1: It is important to refer to the equilibrium, with respect to which the linearisation is done.

2.2 Phase and group velocities

Phase velocity

$$\mathbf{v}_{\mathrm{ph}} = \frac{\omega}{k}\frac{\mathbf{k}}{k}. \tag{2.11}$$

It can be $|\mathbf{v}_{\mathrm{ph}}| > c$, as $\mathbf{v}_{\mathrm{ph}}$ does not carry information.

Group velocity

$$\mathbf{v}_g = \frac{\partial \omega}{\partial \mathbf{k}}. \tag{2.12}$$

It cannot be $|\mathbf{v}_g| > c$, as $\mathbf{v}_g$ does carry information.

2.3 Ideal MHD waves

The ideal MHD system can be reduced by combining its equations, obtaining

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad \rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla p + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} \tag{2.13}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \qquad \frac{\mathrm{d}}{\mathrm{d}t}(p\rho^{-\gamma}) = 0 \quad \text{(see footnote[1])} \tag{2.14}$$

This is a system of 8 equations with 8 unknowns: ρ, p, $\mathbf{u}$, $\mathbf{B}$. We now consider small perturbations to a uniform and static (no flow) equilibrium

$$\mathbf{B}(\mathbf{x}, t) = \mathbf{B}_0 + \mathbf{B}_1(\mathbf{x}, t) \qquad \mathbf{u}(\mathbf{x}, t) = \mathbf{u}_1(\mathbf{x}, t) \tag{2.15}$$

$$\rho(\mathbf{x}, t) = \rho_0 + \rho_1(\mathbf{x}, t) \qquad p(\mathbf{x}, t) = p_0 + p_1(\mathbf{x}, t) \tag{2.16}$$

and linearize the original system of equations concerning the equilibrium

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}_1 = 0 \qquad \rho_0 \frac{\partial \mathbf{u}_1}{\partial t} = -\nabla p_1 + \frac{1}{\mu_0}(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 \tag{2.17}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0) \qquad p_1 = \frac{\gamma p_0}{\rho_0}\rho_1 \equiv c_s^2 \rho_1 \quad \text{(see footnote[2])} \tag{2.18}$$

Here $c_s \equiv \sqrt{\gamma p_0/\rho_0}$ is the sound speed. After the elimination of p_1 and Fourier transformation, this becomes

$$-\omega \rho_1 + \rho_0 \mathbf{k} \cdot \mathbf{u}_1 = 0 \tag{2.19}$$

$$-\omega \rho_0 \mathbf{u}_1 = -\mathbf{k} \rho_1 c_s^2 + \frac{1}{\mu_0}(\mathbf{k} \times \mathbf{B}_1) \times \mathbf{B}_0 \tag{2.20}$$

$$-\omega \mathbf{B}_1 = \mathbf{k} \times (\mathbf{u}_1 \times \mathbf{B}_0) \tag{2.21}$$

[1] Rewriting the continuity equation as $\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho \nabla \cdot \mathbf{u}$, we have another form of the equation of state: $\frac{\mathrm{d}p}{\mathrm{d}t} + \gamma p \nabla \cdot \mathbf{u} = 0$.

[2] From eq.(2.14) and eq.(2.16) we have $(p_0 + p_1)(\rho_0 + \rho_1)^{-\gamma} = p_0 \rho_0^{-\gamma} \Rightarrow (p_0 + p_1)(1 - \gamma \frac{\rho_1}{\rho_0}) = p_0$. At the 'zero' order (i.e. neglecting all the perturbation terms labeled as '1') we simply have $p_0 \equiv p_0$, while at the first order we obtain $p_1 = \gamma p_0 \frac{\rho_1}{\rho_0}$.

The shear Alfvén wave

Without loss of generality we can choose $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ and $k_y = 0$ (see figure 1). Let us now consider the particular case of a transverse wave $u_{1x} = u_{1z} = 0$, i.e.

$$\mathbf{k} = (k_x, 0, k_z) \tag{2.22}$$

$$\mathbf{u}_1 = (0, u_{1y}, 0) \tag{2.23}$$

We will treat the case $u_{1x} \neq 0 \neq u_{1z}$ later.

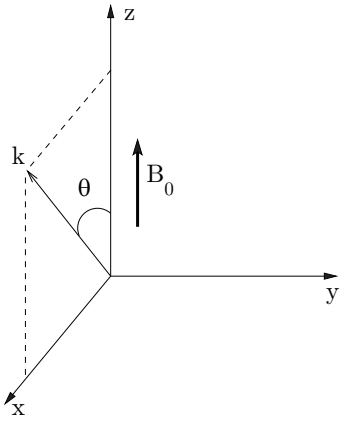

Figure 1: *Notation for the study of MHD waves.*

Eq.(2.19) gives $\begin{pmatrix} k_x \\ 0 \\ k_z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ u_{1y} \\ 0 \end{pmatrix} = 0$

Therefore $\rho_1 = 0$, i.e. there is no variation of the mass density and we can conclude that the wave is of non–compressional type.

The component along the y-axis of eq.(2.20) becomes

$$\omega\rho_0 u_{1y} = -\frac{1}{\mu_0}\Big[(\mathbf{k}\times\mathbf{B}_1)\times\mathbf{B}_0\Big]_y = -\frac{1}{\mu_0}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ (\mathbf{k}\times\mathbf{B}_1)_x & (\mathbf{k}\times\mathbf{B}_1)_y & (\mathbf{k}\times\mathbf{B}_1)_z \\ 0 & 0 & B_0 \end{vmatrix}_y =$$

$$= \frac{B_0}{\mu_0}(\mathbf{k}\times\mathbf{B}_1)_x = \frac{B_0}{\mu_0}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ k_x & 0 & k_z \\ B_{1x} & B_{1y} & B_{1z} \end{vmatrix}_x = -\frac{B_0}{\mu_0}\,k_z B_{1y}$$

Eq.(2.21) gives

$$-\omega B_{1y} = \Big[\mathbf{k}\times(\mathbf{u}_1\times\mathbf{B}_0)\Big]_y = \Big[\mathbf{k}\times\hat{x}B_o u_{1y}\Big]_y = B_0 k_z u_{1y} \tag{2.24}$$

Then the system of eq.(2.19), eq.(2.20) and eq.(2.21) can be written as:

$$\rho_1 = 0, \tag{2.25}$$

$$\omega\rho_0 u_{1y} + \frac{k_z B_0}{\mu_0} B_{1y} = 0, \tag{2.26}$$

$$k_z B_0 u_{1y} + \omega B_{1y} = 0, \tag{2.27}$$

where eq.(2.26) and eq.(2.27) can be written as a homogeneous linear system

$$\mathbf{A}\cdot\begin{pmatrix} u_{1y} \\ B_{1y} \end{pmatrix} = 0, \qquad \text{where} \qquad \mathbf{A} = \begin{pmatrix} \omega\rho_0 & \frac{k_z B_0}{\mu_0} \\ k_z B_0 & \omega \end{pmatrix}. \tag{2.28}$$

To have a non–trivial solution ($u_{1y} \neq 0 \neq B_{1y}$), we must have $\det\mathbf{A} = 0$. Thus, we obtain the following dispersion relation

$$\omega^2 = \frac{B_0^2}{\rho_0 \mu_0} k_z^2 \equiv c_A^2 k_z^2 = c_A^2 k^2 \cos^2\theta, \tag{2.29}$$

where $c_A \equiv B_0/\sqrt{\mu_0 \rho_0}$ is the Alfvén speed. Typical values are

Magnetosphere:

$$\left.\begin{array}{r} B \sim 5 \times 10^{-8}\ \mathrm{T} \\ n \sim 10^6\ \mathrm{m}^{-3} \end{array}\right\} \Rightarrow c_A \sim \frac{5 \times 10^{-8}}{\sqrt{1.7 \times 10^{-27} \cdot 10^6 \cdot 4\pi \cdot 10^{-7}}} \sim 10^6\ \mathrm{m/s}.$$

Tokamak:

$$\left.\begin{array}{r} B \sim 3\ \mathrm{T} \\ n \sim 10^{20}\ \mathrm{m}^{-3} \end{array}\right\} \Rightarrow c_A \sim \frac{3}{\sqrt{1.7 \times 10^{-27} \cdot 10^{20} \cdot 4\pi \cdot 10^{-7}}} \sim 6 \times 10^6\ \mathrm{m/s}.$$

The solution given by eq.(2.29) is called shear Alfvén wave or non–compressional Alfvén wave, as there is no density perturbation:

$$\rho_1 = \frac{\mathbf{k} \cdot \mathbf{u}_1}{\omega} = 0, \tag{2.30}$$

This is different from sound waves, for example. Note that

- The velocity of α particles born with energies 3.5 MeV is $> c_A$, so the α's become resonant[3] with Alfvén waves during slowing down in a fusion reactor.
- Alfvén waves are equivalent to waves on a string with tension S and mass per unit length M

  $$M\frac{\partial^2 y}{\partial t^2} = S\frac{\partial^2 y}{\partial z^2} \qquad \Longrightarrow \qquad \omega^2 = \frac{S}{M}k_z^2 \tag{2.31}$$

  In the exercise, you will show the analogy between a wave travelling along a magnetic field line and a chord.

The compressional Alfvén waves (fast waves) and the magneto–sonic waves

Now we consider the other case $u_{1x} \neq 0$, $u_{1y} = 0$, $u_{1z} \neq 0$, where the perturbation has a longitudinal component. Choosing $B_{1y} = 0$ we get with our previous choices $\mathbf{B}_0 = B_0 \mathbf{e}_z$ and $k_y = 0$ from eq.(2.21)

$$\mathbf{B}_1 = \frac{u_{1x} B_0}{\omega}(\mathbf{k} \times \hat{\mathbf{y}}). \tag{2.32}$$

By inserting ρ_1 from eq.(2.19) and $\mathbf{B}_1$ from eq.(2.32) in eq.(2.20), we get a linear system for u_{1x} and u_{1z}, which again has a non–trivial solution only if the determinant of the coefficient matrix vanishes. After some algebra one finds the dispersion relation

$$\omega^4 - \omega^2 k^2 (c_A^2 + c_s^2) + k_z^2 k^2 c_A^2 c_s^2 = 0, \tag{2.33}$$

[3] As we will see later in the kinetic model, the condition $v_{\mathrm{particle}} \sim v_{\mathrm{ph}}$ makes it possible that a strong interaction between waves and particles with exchange of energy may occur. This may lead to instabilities, and the particle motion may be affected by the wave.

which has the solutions

$$\omega^2 = \frac{1}{2}\left(c_A^2 + c_s^2\right)k^2 \pm \sqrt{\frac{1}{4}\left(c_A^2 + c_s^2\right)^2 k^4 - c_A^2 c_s^2 k^2 k_z^2}. \tag{2.34}$$

Note that

$$\left(\frac{c_s}{c_A}\right)^2 = \frac{\gamma p_0}{\rho_0}\frac{\mu_0 \rho_0}{B_0^2} = \frac{\gamma}{2}\frac{p_0}{\frac{B_0^2}{2\mu_0}} = \frac{\gamma}{2}\beta, \tag{2.35}$$

The pressure ratio β is an important parameter to characterize a plasma[4]. For many plasmas of interest we have $\beta \ll 1$, so $c_s \ll c_A$. In this limit the "+" branch of eq.(2.34) becomes

$$\omega^2 \simeq k^2 c_A^2. \tag{2.36}$$

This solution is called fast wave or compressional Alfvén wave[5]. For the "−" branch we find the so-called slow wave or magneto–sonic wave

$$\omega^2 \simeq c_s^2 k_z^2 = k^2 c_s^2 \cos^2\theta. \tag{2.37}$$

These are all possible modes of oscillation that an (unbounded) "MHD plasma" can sustain. As we relax the assumptions that lead to the MHD model many other modes appear, for example separating ions and electrons in their oscillatory motion. To describe these modes we need a more detailed plasma model, as the multi–fluid or the kinetic models.

[4] $B_0^2/2\mu_0$ is often referred to as "magnetic pressure".
[5] $\rho_1 \neq 0 \longleftrightarrow \nabla \cdot \mathbf{u}_1 \neq 0$