

Solutions to Problem Set 5

Exercise 1 - Two derivations of the continuity equation

Eulerian approach

We are at a fixed point, we choose an observation volume $\Delta V = \Delta x \Delta y \Delta z$. The rate of change of the number of particles in the volume (no sources) is:

$$\frac{\partial}{\partial t} [n \Delta x \Delta y \Delta z] = -\text{Flow out of volume} \times \text{Surface area}$$

Let $\mathbf{v} = \{v_x, v_y, v_z\}$. The flow out of the box times the surface area which they cross is:

$$\{nv_x(\Delta x) \Delta y \Delta z - nv_x(0) \Delta y \Delta z\} + \text{same for } y + \text{same for } z$$

But, since the volume element is infinitesimally small:

$$nv_x(\Delta x) = nv_x(0) + \Delta x \frac{\partial}{\partial x} (nv_x)$$

So the flow multiplied by the surface area which particles cross is given by:

$$\begin{aligned} \text{Flow out} \times \text{Surface area} &= \frac{\partial}{\partial x} (nv_x) \Delta x \Delta y \Delta z + \frac{\partial}{\partial y} (nv_y) \Delta x \Delta y \Delta z + \frac{\partial}{\partial z} (nv_z) \Delta x \Delta y \Delta z \\ &= \nabla \cdot (n\mathbf{v}) \Delta V \end{aligned}$$

so

$$\frac{\partial}{\partial t} [n \Delta V] = -\nabla \cdot (n\mathbf{v}) \Delta V$$

or, since ΔV is fixed

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0$$

Note that this is actually Gauss' theorem $\int_V \nabla \cdot \mathbf{A} dV = \int_S \mathbf{A} \cdot d\mathbf{S}$.

Lagrangian approach

Now we follow the volume. This means that the number of particles in the volume is constant but the volume itself is not constant – the *density* will change.

We take the “Lagrangian” or *total* or *convective* derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$ and apply it to the density (i.e. #particles/volume)

$$\frac{dn}{dt} = \frac{d}{dt} \left(\frac{\Delta N}{\Delta V} \right) = \Delta N \frac{d}{dt} \left(\frac{1}{\Delta V} \right) = -\frac{\Delta N}{\Delta V^2} \frac{d(\Delta V)}{dt} = -n \frac{1}{\Delta V} \frac{d(\Delta V)}{dt}$$

As $\Delta V = \Delta x \Delta y \Delta z$

$$\frac{d(\Delta V)}{dt} = \frac{d\Delta x}{dt} \Delta y \Delta z + \frac{d\Delta y}{dt} \Delta x \Delta z + \frac{d\Delta z}{dt} \Delta x \Delta y = \Delta V \left\{ \frac{1}{\Delta x} \frac{d\Delta x}{dt} + \frac{1}{\Delta y} \frac{d\Delta y}{dt} + \frac{1}{\Delta z} \frac{d\Delta z}{dt} \right\}$$

Now what is $\frac{d(\Delta x)}{dt}$? This is the infinitesimal unit of length due to the non-uniform velocity, which deforms the volume element as it moves in the flow.

$$\frac{d(\Delta x)}{dt} = v_x(\Delta x) - v_x(0) = \Delta x \frac{\partial v_x}{\partial x}$$

so

$$\frac{d(\Delta V)}{dt} = \Delta V \left\{ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right\} = \Delta V (\nabla \cdot \mathbf{v})$$

$$\frac{dn}{dt} = -\frac{n}{\Delta V} \Delta V (\nabla \cdot \mathbf{v}) = -n (\nabla \cdot \mathbf{v})$$

But

$$\begin{aligned} \frac{dn}{dt} &= \frac{\partial n}{\partial t} + (\mathbf{v} \cdot \nabla) n = -n (\nabla \cdot \mathbf{v}) \\ \frac{\partial n}{\partial t} + (\mathbf{v} \cdot \nabla) n + n (\nabla \cdot \mathbf{v}) &= 0 \\ \frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}) &= 0 \end{aligned}$$

The two views, Eulerian and Lagrangian, are equivalent.

Exercise 2 - Magnetic field diffusion in a resistive plasma

a) The resistive MHD equations are:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0 \\ \rho \frac{d\mathbf{u}}{dt} &= \mathbf{J} \times \mathbf{B} - \nabla p \\ \mathbf{E} + \mathbf{u} \times \mathbf{B} &= \eta \mathbf{J} \\ \frac{d}{dt} (p \rho^{-\gamma}) &= 0 \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{J} &= 0 \end{aligned}$$

We are looking for an equation to describe the evolution (in time) of the magnetic field $\mathbf{B}$. It is natural to start from the Faraday equation:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}. \quad (1)$$

We can rewrite $\mathbf{E}$ as a function of $\mathbf{u}$ and $\mathbf{J}$ (Ohm's law) as a function of $\mathbf{B}$ using the Ampère's law:

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta \mathbf{J} = -\mathbf{u} \times \mathbf{B} + \frac{\eta}{\mu_0} \nabla \times \mathbf{B}. \quad (2)$$

From this equation we have:

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = -\frac{\eta}{\mu_0} \nabla \times (\nabla \times \mathbf{B}) \quad (\text{we suppose } \eta \text{ constant})$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = -\frac{\eta}{\mu_0} (\underbrace{\nabla(\nabla \cdot \mathbf{B})}_{=0} - \nabla^2 \mathbf{B})$$

and finally:

$$\frac{\partial \mathbf{B}}{\partial t} - \underbrace{\nabla \times (\mathbf{u} \times \mathbf{B})}_{\text{convection}} - \underbrace{\frac{\eta}{\mu_0} \nabla^2 \mathbf{B}}_{\text{diffusion}} = 0. \quad (3)$$

Equation (3) describes the magnetic field transport in a plasma due to the resistivity arising from the collisions. The term η/μ_0 can be interpreted as a diffusion coefficient (D).

- b) The characteristic time for the diffusion of the magnetic field $\mathbf{B}$ in the plasma can be estimated from $D \sim L^2/\tau$:

$$\tau = \left(\frac{L^2 \mu_0}{\eta} \right)$$

We can estimate the resistivity using the Spitzer's formula: $\eta \simeq 8.72 \times 10^{-10} \Omega \text{ m}$:

$$\tau \simeq \frac{3^2 \cdot 4\pi \cdot 10^{-7}}{8.72 \times 10^{-10}} \simeq 13000 \text{ s (!!!)}$$

From this example we see that for typical discharges in fusion devices (1000 s in ITER) the magnetic field can be considered to be frozen in the plasma.