Nuclear Fusion and Plasma Physics

EPFL

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Solutions to Problem Set 5

Exercise 1 - Two derivations of the continuity equation

Eulerian approach

We are at a fixed point, we choose an observation volume $\Delta V = \Delta x \Delta y \Delta z$. The rate of change of the number of particles in the volume (no sources) is:

$$\frac{\partial}{\partial t} [n\Delta x \Delta y \Delta z] = -\text{Flow out of volume} \times \text{Surface area}$$

Let $\mathbf{v} = \{v_x, v_y, v_z\}$. The flow out of the box times the surface area which they cross is:

$$\{nv_x(\Delta x)\Delta y\Delta z - nv_x(0)\Delta y\Delta z\}$$
 + same for y + same for z

But, since the volume element is infinitesimally small:

$$nv_x(\Delta x) = nv_x(0) + \Delta x \frac{\partial}{\partial x}(nv_x)$$

So the flow multiplied by the surface area which particles cross is given by:

Flow out × Surface area =
$$\frac{\partial}{\partial x}(nv_x)\Delta x\Delta y\Delta z + \frac{\partial}{\partial y}(nv_y)\Delta x\Delta y\Delta z + \frac{\partial}{\partial z}(nv_z)\Delta x\Delta y\Delta z$$

= $\nabla \cdot (n\mathbf{v})\Delta V$

so

$$\frac{\partial}{\partial t} \left[n\Delta V \right] = -\nabla \cdot (n\mathbf{v})\Delta V$$

or, since ΔV is fixed

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0$$

Note that this is actually Gauss' theorem $\int_V \nabla \cdot \mathbf{A} dV = \int_S \mathbf{A} \cdot \mathbf{dS}$.

Lagrangian approach

Now we follow the volume. This means that the number of particles in the volume is constant but the volume itself is not constant – the density will change.

We take the "Lagrangian" or *total* or *convective* derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$ and apply it to the density (i.e. #particles/volume)

$$\frac{dn}{dt} = \frac{d}{dt} \left(\frac{\Delta N}{\Delta V} \right) = \Delta N \frac{d}{dt} \left(\frac{1}{\Delta V} \right) = -\frac{\Delta N}{\Delta V^2} \frac{d \left(\Delta V \right)}{dt} = -n \frac{1}{\Delta V} \frac{d \left(\Delta V \right)}{dt}$$

As $\Delta V = \Delta x \Delta y \Delta z$

$$\frac{d(\Delta V)}{dt} = \frac{d\Delta x}{dt} \Delta y \Delta z + \frac{d\Delta y}{dt} \Delta x \Delta z + \frac{d\Delta z}{dt} \Delta x \Delta y = \Delta V \left\{ \frac{1}{\Delta x} \frac{d\Delta x}{dt} + \frac{1}{\Delta y} \frac{d\Delta y}{dt} + \frac{1}{\Delta z} \frac{d\Delta z}{dt} \right\}$$

Now what is $\frac{d(\Delta x)}{dt}$? This is the infinitesimal unit of length due to the non-uniform velocity, which deforms the volume element as it moves in the flow.

$$\frac{d(\Delta x)}{dt} = v_x(\Delta x) - v_x(0) = \Delta x \frac{\partial v_x}{\partial x}$$

SO

$$\frac{d(\Delta V)}{dt} = \Delta V \left\{ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right\} = \Delta V (\nabla \cdot \mathbf{v})$$
$$\frac{dn}{dt} = -\frac{n}{\Delta V} \Delta V (\nabla \cdot \mathbf{v}) = -n(\nabla \cdot \mathbf{v})$$

But

$$\frac{dn}{dt} = \frac{\partial n}{\partial t} + (\mathbf{v} \cdot \nabla)n = -n(\nabla \cdot \mathbf{v})$$
$$\frac{\partial n}{\partial t} + (\mathbf{v} \cdot \nabla)n + n(\nabla \cdot \mathbf{v}) = 0$$
$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0$$

The two views, Eulerian and Lagrangian, are equivalent.

Exercise 2 - Magnetic field diffusion in a resistive plasma

a) The resistive MHD equations are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} = \mathbf{J} \times \mathbf{B} - \nabla p$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{J}$$

$$\frac{\partial}{\partial t} (p\rho^{-\gamma}) = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{J} = 0$$

We are looking for an equation to describe the evolution (in time) of the magnetic field **B**. It is natural to start from the Faraday equation:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}.\tag{1}$$

We can rewrite \mathbf{E} as a function of \mathbf{u} and \mathbf{J} (Ohm's law) as a function of \mathbf{B} using the Ampère's law:

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta \mathbf{J} = -\mathbf{u} \times \mathbf{B} + \frac{\eta}{\mu_0} \nabla \times \mathbf{B}.$$
 (2)

From this equation we have:

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = -\frac{\eta}{\mu_0} \nabla \times (\nabla \times \mathbf{B}) \quad \text{(we suppose η constant)}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = -\frac{\eta}{\mu_0} \left(\nabla (\underbrace{\nabla \cdot \mathbf{B}}_{=0}) - \nabla^2 \mathbf{B} \right)$$

and finally:

$$\frac{\partial \mathbf{B}}{\partial t} - \underbrace{\nabla \times (\mathbf{u} \times \mathbf{B})}_{\text{convection}} - \underbrace{\frac{\eta}{\mu_0} \nabla^2 \mathbf{B}}_{\text{diffusion}} = 0.$$
 (3)

Equation (3) describes the magnetic field transport in a plasma due to the resistivity arising from the collisions. The term η/μ_0 can be interpreted as a diffusion coefficient (D).

b) The characteristic time for the diffusion of the magnetic field **B** in the plasma can be estimated from $D \sim L^2/\tau$:

$$\tau = \left(\frac{L^2 \mu_0}{\eta}\right)$$

We can estimate the resistivity using the Spitzer's formula: $\eta \simeq 8.72 \times 10^{-10} \,\Omega\,\mathrm{m}$:

$$\tau \simeq \frac{3^2 \cdot 4\pi \cdot 10^{-7}}{8.72 \times 10^{-10}} \simeq 13000 \text{ s (!!!)}$$

From this example we see that for typical discharges in fusion devices (1000 s in ITER) the magnetic field can be considered to be frozen in the plasma.