Nuclear Fusion and Plasma Physics **EPFL**
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Solutions to Problem Set 5

Exercise 1 - Two derivations of the continuity equation

Eulerian approach

We are at a fixed point, we choose an observation volume $\Delta V = \Delta x \Delta y \Delta z$. The rate of change of the number of particles in the volume (no sources) is:

$$\frac{\partial}{\partial t}\left[n\Delta x\Delta y\Delta z\right] = -\text{Flow out of volume} \times \text{Surface area}$$

Let $\mathbf{v} = \{v_x, v_y, v_z\}$. The flow out of the box times the surface area which they cross is:

$$\{nv_x(\Delta x)\Delta y\Delta z - nv_x(0)\Delta y\Delta z\} + \text{same for y} + \text{same for z}$$

But, since the volume element is infinitesimally small:

$$nv_x(\Delta x) = nv_x(0) + \Delta x\frac{\partial}{\partial x}(nv_x)$$

So the flow multiplied by the surface area which particles cross is given by:

$$\begin{aligned}\text{Flow out} \times \text{Surface area} &= \frac{\partial}{\partial x}(nv_x)\Delta x\Delta y\Delta z + \frac{\partial}{\partial y}(nv_y)\Delta x\Delta y\Delta z + \frac{\partial}{\partial z}(nv_z)\Delta x\Delta y\Delta z \\ &= \nabla\cdot(n\mathbf{v})\Delta V\end{aligned}$$

so

$$\frac{\partial}{\partial t}\left[n\Delta V\right] = -\nabla\cdot(n\mathbf{v})\Delta V$$

or, since ΔV is fixed

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{v}) = 0$$

Note that this is actually Gauss' theorem $\int_V \nabla\cdot\mathbf{A}dV = \int_S \mathbf{A}\cdot\mathbf{dS}$.

Lagrangian approach

Now we follow the volume. This means that the number of particles in the volume is constant but the volume itself is not constant – the *density* will change.

We take the "Lagrangian" or *total* or *convective* derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$ and apply it to the density (i.e. #particles/volume)

$$\frac{dn}{dt} = \frac{d}{dt}\left(\frac{\Delta N}{\Delta V}\right) = \Delta N\frac{d}{dt}\left(\frac{1}{\Delta V}\right) = -\frac{\Delta N}{\Delta V^2}\frac{d\left(\Delta V\right)}{dt} = -n\frac{1}{\Delta V}\frac{d(\Delta V)}{dt}$$

As $\Delta V = \Delta x \Delta y \Delta z$

$$\frac{d(\Delta V)}{dt} = \frac{d\Delta x}{dt}\Delta y\Delta z + \frac{d\Delta y}{dt}\Delta x\Delta z + \frac{d\Delta z}{dt}\Delta x\Delta y = \Delta V\left\{\frac{1}{\Delta x}\frac{d\Delta x}{dt} + \frac{1}{\Delta y}\frac{d\Delta y}{dt} + \frac{1}{\Delta z}\frac{d\Delta z}{dt}\right\}$$

Now what is $\frac{d(\Delta x)}{dt}$? This is the infinitesimal unit of length due to the non-uniform velocity, which deforms the volume element as it moves in the flow.

$$\frac{d(\Delta x)}{dt} = v_x(\Delta x) - v_x(0) = \Delta x\frac{\partial v_x}{\partial x}$$

so

$$\frac{d(\Delta V)}{dt} = \Delta V\left\{\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right\} = \Delta V(\nabla\cdot\mathbf{v})$$

$$\frac{dn}{dt} = -\frac{n}{\Delta V}\Delta V(\nabla\cdot\mathbf{v}) = -n(\nabla\cdot\mathbf{v})$$

But

$$\begin{aligned}\frac{dn}{dt} =& \frac{\partial n}{\partial t} + (\mathbf{v}\cdot\nabla)n = -n(\nabla\cdot\mathbf{v})\\ & \frac{\partial n}{\partial t} + (\mathbf{v}\cdot\nabla)n + n(\nabla\cdot\mathbf{v}) = 0\\ & \frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{v}) = 0\end{aligned}$$

The two views, Eulerian and Lagrangian, are equivalent.

Exercise 2 - Magnetic field diffusion in a resistive plasma

a) The resistive MHD equations are:

$$\begin{aligned}\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) &= 0\\ \rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} &= \mathbf{J}\times\mathbf{B} - \nabla p\\ \mathbf{E} + \mathbf{u}\times\mathbf{B} &= \eta\mathbf{J}\\ \frac{\mathrm{d}}{\mathrm{d}t}(p\rho^{-\gamma}) &= 0\\ \nabla\times\mathbf{B} &= \mu_0\mathbf{J}\\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}\\ \nabla\cdot\mathbf{B} &= 0\\ \nabla\cdot\mathbf{J} &= 0\end{aligned}$$

We are looking for an equation to describe the evolution (in time) of the magnetic field $\mathbf{B}$. It is natural to start from the Faraday equation:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}. \tag{1}$$

We can rewrite **E** as a function of **u** and **J** (Ohm's law) as a function of **B** using the Ampère's law:

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta \mathbf{J} = -\mathbf{u} \times \mathbf{B} + \frac{\eta}{\mu_0} \nabla \times \mathbf{B}. \tag{2}$$

From this equation we have:

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = -\frac{\eta}{\mu_0} \nabla \times (\nabla \times \mathbf{B}) \quad \text{(we suppose } \eta \text{ constant)}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = -\frac{\eta}{\mu_0} \big(\nabla (\underbrace{\nabla \cdot \mathbf{B}}_{=0}) - \nabla^2 \mathbf{B} \big)$$

and finally:

$$\frac{\partial \mathbf{B}}{\partial t} - \underbrace{\nabla \times (\mathbf{u} \times \mathbf{B})}_{\text{convection}} - \underbrace{\frac{\eta}{\mu_0} \nabla^2 \mathbf{B}}_{\text{diffusion}} = 0. \tag{3}$$

Equation (3) describes the magnetic field transport in a plasma due to the resistivity arising from the collisions. The term η/μ_0 can be interpreted as a diffusion coefficient (D).

b) The characteristic time for the diffusion of the magnetic field **B** in the plasma can be estimated from $D \sim L^2/\tau$:

$$\tau = \left(\frac{L^2 \mu_0}{\eta} \right)$$

We can estimate the resistivity using the Spitzer's formula: $\eta \simeq 8.72 \times 10^{-10}\,\Omega\,\mathrm{m}$:

$$\tau \simeq \frac{3^2 \cdot 4\pi \cdot 10^{-7}}{8.72 \times 10^{-10}} \simeq 13000 \text{ s (!!!)}$$

From this example we see that for typical discharges in fusion devices (1000 s in ITER) the magnetic field can be considered to be frozen in the plasma.