Nuclear Fusion and Plasma Physics

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Fluid description of plasmas

- Fluid description of plasmas
  - general hierarchy of plasma models
  - fluid quantities
  - Eulerian and Lagrangian approaches
- Two fluid model
  - basic equations
- Single fluid model
  - new variables
  - combination of two-fluid equations
  - approximations: slow motion, large scale, quasi-neutrality
- The MHD model

1 Fluid description of plasmas

Memento: plasma self-consistency

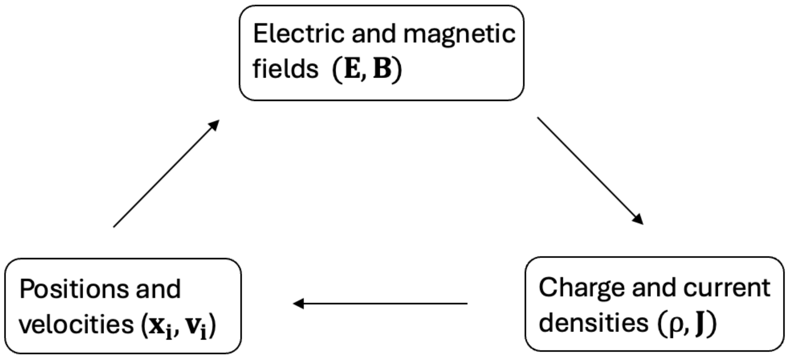

Figure 1: Illustration of plasma self-consistency.

Models for plasmas:

1. single-particle
2. kinetic (Boltzmann equation)
3. multi-fluid
4. single fluid

Formally one could start from 1., then introduce the concept of distribution function and get to 2., then average over the distribution (different "moments") to get to 3., and then "average" over different species to get to 4.

- We have seen particle orbits in 1., and identified guiding center drifts,
- We have calculated exchanges of energy and momentum via collisions to "obtain" equilibrium distributions.
- Now we take a *macroscopic approach* and describe the plasma as a fluid.

A charged fluid is described by:

- **Particle number density:** $n_{e,i} = n_{e,i}(\mathbf{x}, t)$

- **Charge density:** $\rho(\mathbf{x}, t) = \sum_j n_j q_j = e(Z_i n_i - n_e)$ (j indicates the different species).

- **Current density:** $\mathbf{J}(\mathbf{x}, t) = \sum_j n_j q_j \mathbf{v}_j = e(Z_i n_i \mathbf{v}_i - n_e \mathbf{v}_e) \cong e n_e(\mathbf{v}_i - \mathbf{v}_e)$. $[Z_i = 1]$

- **Velocity field:** $\mathbf{v}_j = \mathbf{v}_j(\mathbf{x}, t)$, flux $\mathbf{\Gamma}_j = n_j \mathbf{v}_j$.

These quantities must be considered as averages of all the individual particles that form the fluid element at $\mathbf{x}$. E.g.

$$\mathbf{v} = \sum_{\text{particles in } \Delta V} \frac{\mathbf{u}_i}{\Delta V n} = \sum_{\text{particles in } \Delta V} \frac{\mathbf{u}_i}{\Delta N}$$

where $\Delta N = \Delta V n$ is the number of particles inside ΔV. The differential fluid element $\mathrm{d}\mathbf{x}$ should be considered as 'microscopically big": i.e., the result of an average of many particles, and "macroscopically small" i.e., for which can apply differential calculus.

All the quantities involved in the fluid description should be considered as the result of an average also over velocity space.

1.1 Two ways of describing fluid dynamics: Lagrangian and Eulerian

1. Lagrangian: we follow a fluid element in its evolution ("we sit on top of it"), i.e., the exact number of particles that form the initial element of volume ΔV.

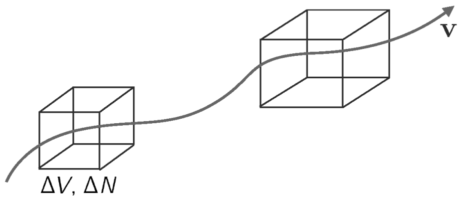

Figure 2: Lagrangian approach in fluid dynamics.

Variations are seen because we move within the fluid and because each point varies in time. The so-called *Lagrangian* or *total* or *convective* derivative is:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

Example:
$G = G(x, t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t} G(x, t) = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial G}{\partial t} + v_x \frac{\partial G}{\partial x}.$$

2. Eulerian: the observer stays at a fixed point in space. The fluid goes through the *observed volume element*.

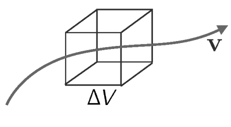

Figure 3: Eulerian approach in fluid dynamics.

In the Eulerian view, the only possibility to have time variations is to have an *explicit* time dependence $\frac{\partial}{\partial t}$.

**See exercise series for more details*

2 Two fluid model

2.1 Continuity equation

We have the first equation from the conservation of mass,

$$\boxed{\frac{\partial n_{e,i}}{\partial t} + \nabla \cdot (n_{e,i}\, \mathbf{v}_{e,i}) = 0,}$$

which is valid for each species. To make the notation less heavy, we will drop the subscripts e, i from here onwards, where this does not create any confusion.
See the exercise series for a derivation of the continuity equation in both the Lagrangian and the Eulerian approaches.

2.2 Equation of motion

The second equation is for *conservation of momentum* (equation of motion). We use the Lagrangian view for the differential volume element of fluid.

This description assumes the presence of three forces: the **Lorentz force** due to electromagnetic fields, a **force resulting from pressure gradients**, and a **force due to collisional drag**.

Lorentz force

First, let's consider Lorentz force on volume ΔV (for one species)

$$\mathbf{F}_{tot} = \sum_{j\,\text{(particles)}} q_j\,(\mathbf{E} + \mathbf{u}_j \times \mathbf{B}) = \Delta N\, q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad \text{as } \mathbf{v} = \frac{\sum_j \mathbf{u}_j}{\Delta N} \qquad \text{and } q = \frac{\sum_j q_j}{\Delta N}$$

Thus, the force per unit volume is:

$$\frac{\mathbf{F}_{tot}}{\Delta V} = \frac{\Delta N}{\Delta V}\, q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = n\, q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Pressure

Let's evaluate the pressure force. For this, we consider that particles move in and out of the volume element due to their random motion. Note that on *average* the number of particles in the fluid element does not vary (remember: we are considering the Lagrangian view).

p is the force per unit area due to thermal motion ($p = nT$), felt by the fluid element.

Example: Isotropic pressure

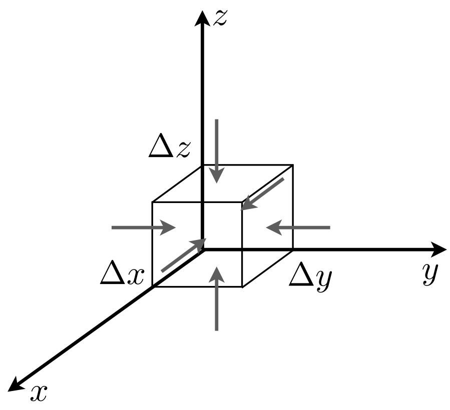

Figure 4: Isotropic pressure on a fluid element.

Net force (for example) in x-direction:

$$F_x = \big(p(0) - p(\Delta x)\big)\Delta y \Delta z \cong -\frac{\partial p}{\partial x}\Delta x \Delta y \Delta z.$$

So,

$$F_x = -\frac{\partial p}{\partial x}\Delta V, \text{ and analogously } F_y = -\frac{\partial p}{\partial y}\Delta V \qquad F_z = -\frac{\partial p}{\partial z}\Delta V$$

We obtain:

$$\frac{\mathbf{F}}{\Delta V} = -\nabla p.$$

Note 5.2.1: In the Lagrangian view there is no net change in the number of particles in the volume element. However, particles crossing the boundary in and out can indeed exchange momentum.

In general, the pressure could be anisotropic

$$\underline{\underline{p}} = \begin{pmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{pmatrix}; \text{ in this case, } \frac{\mathbf{F}}{\Delta V} = -\nabla \cdot \underline{\underline{p}}$$

Collisional drag

The exchange of momentum between the different species (electron/ions) is given by:

$$\begin{cases} -m_e n_e \bar{\nu}_p^{e/i}(\mathbf{v}_e - \mathbf{v}_i) & \text{Electrons} \\ -m_i n_i \bar{\nu}_p^{i/e}(\mathbf{v}_i - \mathbf{v}_e) & \text{Ions} \end{cases}$$

Equation of conservation of momentum

Momentum per unit volume:

$$\frac{\sum_j m_j \mathbf{u}_j}{\Delta V} = m\frac{\Delta N}{\Delta V}\mathbf{v} = m\,n\,\mathbf{v}. \qquad \text{For one species}$$

$$\text{“Force/volume”} = n\,q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \text{“pressure force”} + \text{“collisional drag”}$$

In our Lagrangian picture (in which we maintain the same number of particles):

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{momentum in } \Delta V) = \Delta V n_{e,i} q_{e,i} (\mathbf{E} + \mathbf{v}_{e,i} \times \mathbf{B}) - \Delta V \nabla p_{e,i} - \Delta V m_{e,i} n_{e,i} \bar{\nu}_p^{e/i,i/e} (\mathbf{v}_{e,i} - \mathbf{v}_{i,e})$$

Momentum in ΔV is given by $m_{e,i}\, n_{e,i}\, \Delta V \mathbf{v}_{e,i}$. As $n_{e,i} \Delta V = \Delta N$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_{e,i}\, n_{e,i}\, \Delta V \mathbf{v}_{e,i}) = m_{e,i} \frac{\mathrm{d}}{\mathrm{d}t}(\Delta N\, \mathbf{v}_{e_i}) = \Delta N m_{e_i} \frac{\mathrm{d}\mathbf{v}_{e_i}}{\mathrm{d}t} = n_{e,i} \Delta V m_{e_i} \frac{\mathrm{d}\mathbf{v}_{e_i}}{\mathrm{d}t}$$

where we have used the fact that the number of particles in fluid volume is constant by the definition of the Lagrangian view.

We can divide the whole equation by ΔV and obtain the equation of conservation of momentum,

$$\boxed{m_{e,i} n_{e,i} \frac{\mathrm{d}\mathbf{v}_{e,i}}{\mathrm{d}t} = q_{e,i} n_{e,i} (\mathbf{E} + \mathbf{v}_{e_i} \times \mathbf{B}) - \nabla p_{e,i} - m_{e,i} n_{e,i} \bar{\nu}_p^{e/i,i/e} (\mathbf{v}_{e,i} - \mathbf{v}_{i,e}).}$$

2.3 Equation of state

How to treat the pressure p? We already assumed a scalar p. To fully describe the pressure, we would need to consider the equation of conservation of energy, but this will include the heat flux. In general, the equation for the n^{th} order moment contains the $(n+1)^{th}$ moment of the distribution.

Example:

Continuity (0^{th} order)	contains $\mathbf{v}$ (1^{st} order), determined by:
Momentum (1^{st} order)	contains p (2^{nd} order), determined by:
Energy (2^{nd} order)	contains $\mathbf{Q}$ (3^{rd} order), determined by:
....	

Similarly, higher-order moments of the distribution function will involve even higher-order terms.

For example, the heat flux $\mathbf{Q}$ is given by:

$$\boxed{\mathbf{Q} = mn \int \mathrm{d}\mathbf{u}\ \mathbf{u}\ f(\mathbf{u})\, |\mathbf{u} - \mathbf{u}_0|^2}$$

To truncate the “hierarchy”, we can consider a thermodynamic equation of state for the plasma.

$$\boxed{pV^{\gamma} = constant},$$

where γ is related to the assumption on the property of the heat flux:

1. Isothermal compression: p increases only because density increases: $\gamma = 1$. Isothermal transformations are possible in a real plasma, for example, along $\mathbf{B}$: fast particle streaming along $\mathbf{B}$ can maintain a constant temperature (ex. plasma wave along $\mathbf{B}$).

2. Adiabatic process: compression must be faster than heat exchange. Possible across $\mathbf{B}$,

$$\gamma = \frac{c_p}{c_V} = \frac{N_{dof} + 2}{N_{dof}}$$

Where N_{dof} is the number of degrees of freedom. Note that γ is often referred to as "*adiabatic exponent*" in both cases.

2.4 Summary of "fluid" equations for a two-species plasma

Continuity equation for both electrons (e) and ions (i):

$$\frac{\partial n_{e,i}}{\partial t} + \nabla \cdot (n_{e,i}\mathbf{v}_{e,i}) = 0 \tag{2.1}$$

Moment equation for both electrons (e) and ions (i):

$$m_{e,i}n_{e,i}\frac{\mathrm{d}\mathbf{v}_{e,i}}{\mathrm{d}t} = q_{e,i}n_{e,i}(\mathbf{E} + \mathbf{v}_{e,i} \times \mathbf{B}) - \nabla p_{e,i} - \underbrace{m_{e,i}n_{e,i}\bar{\nu}_p^{e/i}(\mathbf{v}_{e,i} - \mathbf{v}_{i,e})}_{\mathrm{R}_{e,i}} \tag{2.2}$$

Equation of State:

$$p_{e,i}n_{e,i}^{-\gamma} = \text{ constant} \tag{2.3}$$

Gauss's Law for Electric Field:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \rho = \sum_{e,i} q_{e,i}n_{e,i} = Zen_i - en_e \tag{2.4}$$

Gauss's Law for Magnetic Field:

$$\nabla \cdot \mathbf{B} = 0 \tag{2.5}$$

Faraday's Law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.6}$$

Ampère's Law (with Maxwell's correction):

$$\nabla \times \mathbf{B} = \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad \mathbf{J} = \sum_{e,i} n_{e,i}q_{e,i}\mathbf{v}_{e,i} \tag{2.7}$$

We have 18 equations, but two of Maxwell's equations are redundant. Indeed:

$$\begin{cases} \nabla \cdot \mathbf{B} = 0 \iff \nabla \cdot (\nabla \times \mathbf{E}) = 0 = -\dfrac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) \\ \nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0} \iff \nabla \cdot (\nabla \times \mathbf{B}) = 0 = \mu_0 \nabla \cdot \mathbf{J} + \dfrac{1}{c^2}\dfrac{\partial}{\partial t}(\nabla \cdot \mathbf{E}) \end{cases} \tag{2.8}$$

But,

$$\begin{aligned}\mu_0 \nabla \cdot \mathbf{J} &= \mu_0 \nabla \cdot (\sum_{e,i} q_{e,i} n_{e,i} \mathbf{v}_{e,i}) = \mu_0 \sum_{e,i} q_{e,i} \nabla \cdot (n_{e,i} \mathbf{v}_{e,i}) \\ (continuity) &= -\mu_0 \sum_{e,i} q_{e,i} \frac{\partial n_{e,i}}{\partial t} = -\frac{\partial}{\partial t} \sum_{e,i} \frac{n_{e,i} q_{e_i}}{\varepsilon_0 c^2} \\ &= -\frac{1}{c^2} \frac{\partial}{\partial t} \frac{\rho}{\varepsilon_0}\end{aligned}$$

So,

$$\boxed{0 = \mu_0 \nabla \cdot \mathbf{J} + \frac{1}{c^2} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}) = -\frac{1}{c^2} \frac{\partial}{\partial t} \left(\frac{\rho}{\varepsilon_0} - \nabla \cdot \mathbf{E} \right)}$$

Thus, we have 16 independent equations. How many variables?

$$n_{e,i} \to 2, \mathbf{v}_{e,i} \to 6, p_{e,i} \to 2, \mathbf{E} \to 3, \mathbf{B} \to 3.$$

We have 16 variables! This system of equations is complete, but very complex (e.g. it is nonlinear) and difficult to treat in actual geometry.

To be able to deal with actual plasma configurations (for ex. to find equilibrium states and evaluate their stability) we should simplify the description and, by combining the two-fluid variables and equations, construct a single fluid model.

3 Single fluid model

Let us start by defining the variables that describe a single fluid:

1. Mass density: $\rho_m = m_i n_i + m_e n_e \cong n(m_i + m_e) \cong n m_i$. For simplicity, we work with $Z = 1$. Due to quasi-neutrality, $n_e \cong n_i$.
2. Charge density: $\rho_q = (n_i - n_e)e$ (small but not necessarily zero)
3. Center of mass velocity:

$$\mathbf{u} = \frac{m_i n_i \mathbf{v}_i + m_e n_e \mathbf{v}_e}{m_i n_i + m_e n_e} \cong \mathbf{v}_i + \frac{m_e}{m_i} \mathbf{v}_e \cong \mathbf{v}_i$$

4. Current density: $\mathbf{J} = ne(\mathbf{v}_i - \mathbf{v}_e)$
5. Total pressure: $p = p_e + p_i$, as $u \sim v_i$ and $v_{th,e} \gg u, v_i, v_e$.

The idea is to obtain single-fluid equations from linear combinations of two-fluid equations. Specifically, we will exploit the continuity and momentum equations for electrons and ions.

a) Continuity equations:

$$m_eC_e + m_iC_i \quad \Rightarrow \quad \boxed{\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u}) = 0} \quad \text{mass continuity} \tag{3.1}$$

$$q_eC_e + q_iC_i \quad \Rightarrow \quad \boxed{\frac{\partial \rho_q}{\partial t} + \nabla \cdot \mathbf{J} = 0} \quad \text{charge continuity} \tag{3.2}$$

b) Momentum equations:

To derive the momentum equation for a single fluid, we start by combining the momentum equations for electrons (M_e) and ions (M_i).

$$M_e + M_i \quad \Rightarrow \quad \boxed{\rho_m \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \rho_q \mathbf{E} + \mathbf{J} \times \mathbf{B} - \nabla p} \tag{3.3}$$

Here, ρ_m is the mass density, $\mathbf{u}$ is the center of mass velocity, ρ_q is the charge density, $\mathbf{J}$ is the current density, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, and p is the pressure. The term $\rho_q \mathbf{E} + \mathbf{J} \times \mathbf{B} - \nabla p$ represents the forces acting on the fluid, including the electric force, the Lorentz force, and the pressure gradient force.

Since $\mathbf{R}_{e,i} + \mathbf{R}_{i,e} = 0$, the collisional effects between electrons and ions cancel out in the combined momentum equation. However, we still need a second equation to fully describe the momentum.

We consider the momentum equation for electrons alone:

$$M_e \quad \Rightarrow \quad m_e n_e \frac{\mathrm{d}\mathbf{v}_e}{\mathrm{d}t} = -en_e(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - \nabla p_e - \mathbf{R}_{e,i} \tag{3.4}$$

In this equation, m_e is the mass of an electron, n_e is the electron density, $\mathbf{v}_e$ is the electron velocity, p_e is the electron pressure, and $\mathbf{R}_{e,i}$ is the frictional force between electrons and ions.

Next, we need to express $\mathbf{R}_{e,i}$ in terms of known quantities:

$$\mathbf{R}_{e,i} = m_e n_e \bar{\nu}_p^{e/i} (\mathbf{v}_e - \mathbf{v}_i) = e^2 n_e^2 \eta (\mathbf{v}_e - \mathbf{v}_i) = -en_e \eta \mathbf{J} \tag{3.5}$$

Here, $\bar{\nu}_p^{e/i}$ is the average collision frequency between electrons and ions, and η is the resistivity given by:

$$\eta = \frac{m_e \bar{\nu}_p^{e/i}}{e^2 n_e} \tag{3.6}$$

In this expression, η represents the resistivity of the plasma, which is a measure of how the collisions between particles impede the current. The term $\mathbf{R}_{e,i}$ describes the frictional force between electrons and ions, and by substituting this back into the electron momentum equation, we can understand the effects of collisions on electron motion.

Approximation: Neglect electron inertia: $m_e \to 0$. This means we describe slow phenomena compared to the electron response time.

$$0 \cong -en_e(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - \nabla p_e + en_e\eta\mathbf{J} \quad \Rightarrow \quad \mathbf{E} + \mathbf{v}_e \times \mathbf{B} = \eta\mathbf{J} - \frac{1}{en_e}\nabla p_e.$$

But $\mathbf{v}_e$ can be expressed in terms of $\mathbf{J}$: $\mathbf{J} = en_e(\mathbf{v}_i - \mathbf{v}_e)$, $\mathbf{v}_e = \mathbf{v}_i - \frac{\mathbf{J}}{en_e} \cong \mathbf{u} - \frac{\mathbf{J}}{en_e}$.

Thus, we obtain:

$$\mathbf{E} + \left(\mathbf{u} - \frac{\mathbf{J}}{en_e}\right) \times \mathbf{B} = \eta\mathbf{J} - \frac{1}{en_e}\nabla p_e$$

or

$$\boxed{\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta\mathbf{J} + \frac{\mathbf{J} \times \mathbf{B} - \nabla p_e}{en_e}.} \tag{3.7}$$

The system obtained so far is:

$$\begin{cases} \frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u}) = 0, \quad \frac{\partial \rho_q}{\partial t} + \nabla \cdot \mathbf{J} = 0 & \text{“Continuity” (of mass and charge)} \\ \rho_m \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \rho_q \mathbf{E} + \mathbf{J} \times \mathbf{B} - \nabla p & \text{“Force equation”} \\ \mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta\mathbf{J} + \frac{1}{en_e}(\mathbf{J} \times \mathbf{B} - \nabla p_e) & \text{“Ohm's law”} \\ \frac{\mathrm{d}}{\mathrm{d}t}(p\rho_m^{-\gamma}) = 0 & \text{“State equation”} \\ + \text{ Maxwell's equations} & \end{cases} \tag{3.8}$$

We would like to simplify the system even further. What can we do? We can consider:

- quasi neutrality, $\rho_q \to 0$
- slow phenomena, $\frac{\partial}{\partial t} \to 0$
- large scale phenomena, $\rho_L \ll L$ (L is the plasma scale length)

Approximation:

- **Quasi neutrality:** $\rho_q \sim 0$ except in Poisson's equation to determine $\mathbf{E}$. This assumption, often referred to as “the plasma approximation”, simplifies the Maxwell equations by eliminating the charge density term outside of contexts where electric fields are directly calculated.
- **Slow phenomena:** $\frac{1}{c^2}\frac{\partial \mathrm{E}}{\partial t} \to 0$, leading to $\nabla \times \mathbf{B} \cong \mu_0 \mathbf{J}$. This approximation is valid when electromagnetic phenomena evolve much slower than the speed of light, effectively simplifying Ampère's law by neglecting the displacement current.
- **Large scale phenomena:** Relative to the Larmor radius, this leads to neglecting $\frac{1}{e\,n_e}(\mathbf{J} \times \mathbf{B})$ and $\frac{\nabla p_e}{e\,n_e}$ in Ohm's law. This assumption is crucial for considering dynamics on scales much larger than the microscopic interactions, simplifying the force terms in the generalized Ohm's law.

Proof: we will express the gradient in terms of order of magnitude of characteristic lengths.

$$\frac{\left|\frac{\nabla p_e}{n_e e}\right|}{|\mathbf{u}\times\mathbf{B}|} \underset{T_e=const}{\overset{p_e=T_e n_e}{\sim}} \frac{T_e \nabla n_e}{e n_e u B} \overset{|\nabla n_e|\sim\frac{n_e}{L}}{\sim} \frac{T_e n_e}{L e n_e u B} \overset{\Omega_{c,i}=\frac{eB}{m_i}}{=} \frac{T_e}{L u m_i \Omega_{c,i}} \overset{T_e\sim T_i}{\sim} \frac{T_i}{m_i}\frac{1}{L u \Omega_{c,i}},$$

where L is the scale length. As $v_{th,i}^2 = \frac{T_i}{m_i}$, we obtain:

$$\frac{v_{th,i}^2}{L u \Omega_{c,i}} = \frac{v_{th,i}}{\Omega_{c,i}}\frac{v_{th_i}}{uL} \overset{\rho_{L,i}=\frac{v_{th_i}}{\Omega_{c,i}}}{=} \frac{\rho_{L,i}}{L}\frac{v_{th_i}}{u} \ll 1,$$

as $u \gtrsim v_{th,i}$, and, more importantly, $\rho_{L,i} \ll L$.

For the $\frac{1}{en_e}(\mathbf{J}\times\mathbf{B})$ term one can argue that, because of the equation of motion, considered for $\rho_q \to 0$, and for slow phenomena ($\frac{d}{dt} \to 0$), it is of the same order as the pressure term. So, $|\mathbf{J}\times\mathbf{B}| \approx |\nabla p|$ and both can be neglected in Ohm's law for large-scale phenomena.

3.1 Summary: "MagnetoHydroDynamic" model (MHD)

$$\begin{cases} \frac{\partial\rho_m}{\partial t} + \nabla\cdot(\rho_m\mathbf{u}) = 0 \\ \nabla\cdot\mathbf{J} = 0 \\ \rho_m \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \mathbf{J}\times\mathbf{B} - \nabla p \\ \mathbf{E} + \mathbf{u}\times\mathbf{B} = \begin{cases} \eta\mathbf{J} & \text{“resistive MHD”} \\ 0 & \text{“ideal MHD” (for hot plasmas)} \end{cases} \\ \frac{\mathrm{d}}{\mathrm{d}t}(p\rho_m^{-\gamma}) = 0 \\ \nabla\times\mathbf{B} = \mu_0\mathbf{J} \qquad \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \end{cases} \tag{3.9}$$

Note 3.1.1: $\mathbf{E}+\mathbf{u}\times\mathbf{B}$ correspond to the electric field seen by moving charges. We have 16 equations of which two are redundant, thus leaving us with 14 equations and therefore 14 unknowns (ρ_m, $\mathbf{J}$, p, $\mathbf{u}$, $\mathbf{E}$, $\mathbf{B}$).

This is useful for:

- macroscopic phenomena ($\rho_L \ll L$)
- relatively slow phenomena ($\tau \gg \Omega_{c,i}^{-1}$, $m_e\frac{d}{dt} \to 0$, $\frac{1}{c^2}\frac{\partial E}{\partial t} \to 0$)

3.2 Consequences of ideal MHD model

Magnetic flux is conserved, we say that the flux is frozen into the plasma. The field lines and the flux tubes associated with them acquire an important meaning as if they were real objects. An interesting application of this is the *dynamo effect*.

Qualitative discussion of dynamo effect

The freezing of **B** in plasma is believed to be at the origin of the magnetic field in the universe and in the (melted) metallic core of planets such as the earth through the "dynamo effect", illustrated in figure 5.

The dynamo algorithm starts with first stretching a closed flux rope of cross-section S_0 to twice its length preserving its volume, as in an incompressible flow, see (a) $\rightarrow$ (b) in figure 5. The rope's cross-section then decreases by a factor of two ($S_1 = S_0/2$), and because of flux freezing the magnetic field doubles ($B_1 = 2B_0$). In the next step, the rope is twisted into a figure eight, (b) $\rightarrow$ (c), and then folded, (c) $\rightarrow$ (d), so that now there are two loops, whose fields now point in the same direction and together occupy the same volume as the original flux loop. The flux through this volume has now doubled. The last important step consists of merging the two loops into one, (d) $\rightarrow$ (a), through small diffusive effects. This is important in order that the new arrangement doesn't easily undo itself and the whole process becomes irreversible. The newly merged loops now become topologically the same as the original single loop, but with the field strength scaled up by a factor of 2.

It is believed that complex fluid motion can lead to effective stretching and folding of flux tubes, therefore to amplification (or creation from thermal noise) of magnetic fields ("dynamo effect").

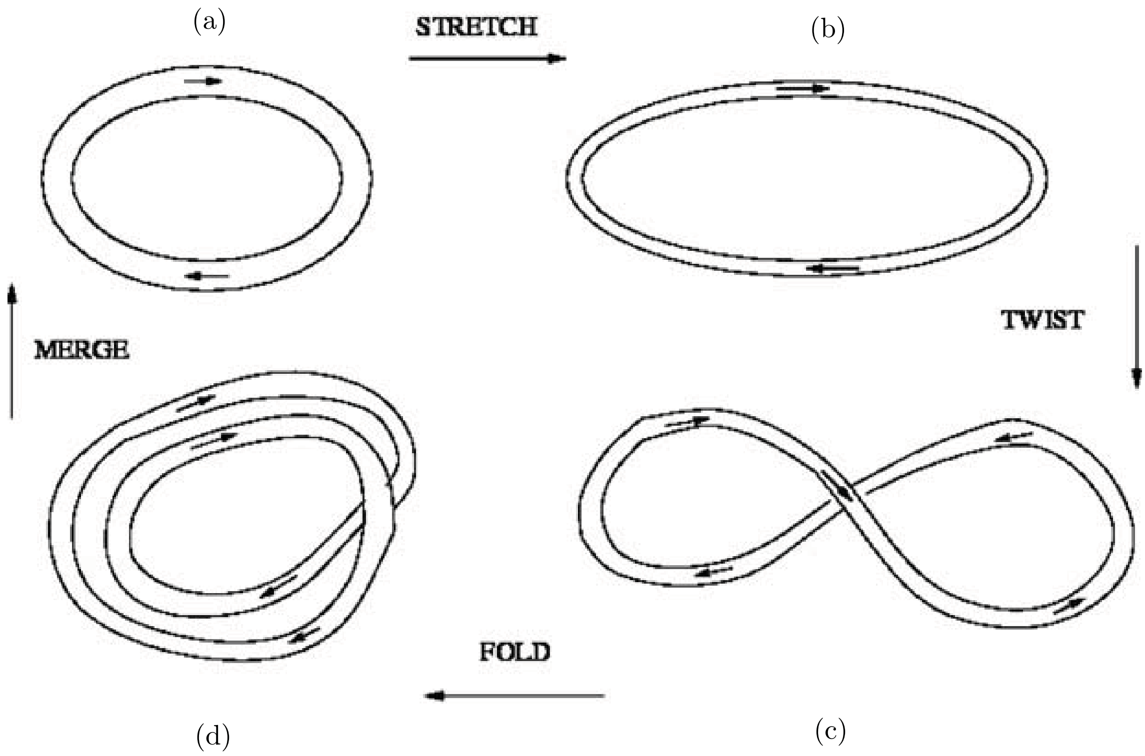

Figure 5: The stretch-twist-fold scheme for fast dynamo effect. *Courtesy of A. Brandenburg, K. Subramanian, Physics Reports 417 (2005) 1–209.*

Another aspect of the flux freezing is the question of solar flares, CME, and their "connection" to our ionosphere and atmosphere. Naturally, field lines and plasma do not stay "frozen" together for an infinite time if the resistivity is finite.

In an exercise you will show that magnetic field follows a diffusion equation, with a diffusion time given by:

$$\boxed{\tau \sim \frac{\mu_0 L^2}{\eta}, \text{ so for } \eta \to 0, \tau \to \infty.} \tag{3.10}$$