

Solutions to Problem Set 4

Exercise 1 - Small vs. large collision angles

a) For the collisions with small deflection angle, the momentum transfer cross-section is:

$$\begin{aligned}\sigma_p &= \sigma_{Ek} \frac{m_1 + m_2}{2m_1} \\ &= \underbrace{8\pi b_{90}^2 \ln \Lambda}_{\sigma_{Ek}} \frac{m_1 m_2}{(m_1 + m_2)^2} \frac{m_1 + m_2}{2m_1} \\ &= 4\pi b_{90}^2 \ln \Lambda \frac{m_2}{m_1 + m_2}\end{aligned}\quad (1)$$

The cross-section σ_p has to be compared with the cross-section for the collisions with a deflection angle $\theta \geq 90^\circ$:

$$\sigma_{90} = \pi b_{90}^2$$

The ratio between them is:

$$\frac{\sigma_p}{\sigma_{90}} = 4 \ln \Lambda \frac{m_2}{m_1 + m_2}$$

and for $m_2 \gg m_1$ (ie electrons colliding with nucleus of deuterium or tritium):

$$\frac{\sigma_p}{\sigma_{90}} \approx 4 \ln \Lambda$$

If $T_e = T_i = 10$ keV the coulomb logarithm is (formulary NRL, page 34)

$$\ln \Lambda = 24 - \ln \left(\frac{\sqrt{n_e [cm^{-3}]}}{T [eV]} \right) \approx 24 - \ln \left(\frac{\sqrt{10^{14}}}{10^4} \right) \approx 17$$

Therefore $\frac{\sigma_p}{\sigma_{90}} \approx 68 \gg 1$.

b) In general, for $m_2 \gg m_1$ and a typical temperature of a plasma used in thermonuclear fusion applications, we have $\frac{\sigma_p}{\sigma_{90}} \gg 1$. We can then conclude that it is possible to neglect the effects of the collisions with $\theta \geq 90^\circ$.

Exercise 2 - Alpha particle thermalisation in a burning plasma

The α particles will collide with the three species (electrons, deuterium ions and tritium ions) losing their energy and therefore their velocity. The estimated relaxation time is necessarily a function of the α 's velocity.

Using the given parameters, we can obtain the relevant thermal speeds:

$$\begin{aligned} v_{\text{th},e} &= \sqrt{\frac{T_e}{m_e}} = 4.19 \times 10^7 \text{ m/s} \\ v_{\text{th},D} &= \sqrt{\frac{T_D}{2 m_p}} = 6.92 \times 10^5 \text{ m/s} \\ v_{\text{th},T} &= \sqrt{\frac{T_T}{3 m_p}} = 5.65 \times 10^5 \text{ m/s} \end{aligned}$$

The α 's velocity is obtained from their energy:

$$v_\alpha = \sqrt{\frac{2 E_\alpha}{4 m_p}} = 6.92 \times 10^3 \sqrt{E_\alpha [\text{eV}]} \text{ m/s}$$

At birth, $E_\alpha = 3.5 \text{ MeV}$, $v_\alpha = 1.3 \times 10^7 \text{ m/s}$ and therefore:

$$v_{\text{th},D} \leq v_{\text{th},T} \ll v_\alpha < v_{\text{th},e}$$

The last relation is valid for $E_\alpha > 10 \text{ keV}$. For $7 \text{ keV} \leq E_\alpha \leq 10 \text{ keV}$, $v_{\text{th},T} \leq v_\alpha \leq v_{\text{th},D}$ and for $E_\alpha \leq 7 \text{ keV}$, we have $v_\alpha \leq v_{\text{th},T}$. When $E_\alpha \approx 5 \text{ keV}$, the α 's are decelerated to a velocity corresponding to $T_\alpha = 10 \text{ keV}$ (complete thermalization).

The three collision frequencies taking part in the energy transfer process are:

$$\begin{aligned} \nu_{E_k}^{\alpha/e} &\simeq n_e \frac{4e^4}{2\pi\epsilon_0^2} \frac{\ln \Lambda}{4 m_p m_e} \frac{1}{v_{\text{the}}^3} \approx 20.4 \text{ s}^{-1} \\ \nu_{E_k}^{\alpha/D} &\simeq \frac{n_e}{2} \frac{4e^4}{2\pi\epsilon_0^2} \frac{\ln \Lambda}{4 m_p 2 m_p} \frac{1}{v_\alpha^3} = \nu_{E_k}^{\alpha/e} \sqrt{\frac{m_p}{2 m_e}} \left(\frac{T}{E_\alpha} \right)^{3/2} \\ \nu_{E_k}^{\alpha/T} &= \frac{2}{3} \nu_E^{\alpha/D} \end{aligned}$$

where we have used the relations:

$$\ln \Lambda = 24 - \ln \left(\frac{\sqrt{n_e [\text{cm}^{-3}]}}{T [\text{eV}]} \right) \approx 24 - \ln \left(\frac{\sqrt{10^{14}}}{10^4} \right) \approx 17.1$$

and

$$n_D = n_T = 0.5 n_e$$

Strictly speaking, the velocity in the denominator is relative to the centre-of-mass reference frame. Usually, this relative velocity is dominated by the velocity of one particle (target particle or on-coming particle). That is the case for the three interactions studied and that is why the electron thermal speed (target particles) is present in the expression of $\nu_{E_k}^{\alpha/e}$ ($v_{\text{th},e} \gg v_\alpha$).

At the beginning of the relaxation process ($E_\alpha = 3.5 \text{ MeV}$), the frequencies are equal to:

$$\nu_{E_k}^{\alpha/D} \approx 9.4 \times 10^{-2} \text{ s}^{-1} \quad \nu_{E_k}^{\alpha/T} \approx 6.3 \times 10^{-2} \text{ s}^{-1}$$

therefore $\nu_{E_k}^{\alpha/D}, \nu_{E_k}^{\alpha/T} \ll \nu_{E_k}^{\alpha/e} \approx 20.4 \text{ s}^{-1}$: the α particles transmit their energy to the electrons.

In this case, the relaxation time is given by:

$$\tau = \left(\nu_{E_k}^{\alpha/D} + \nu_{E_k}^{\alpha/T} + \nu_{E_k}^{\alpha/e} \right)^{-1} \approx \left(\nu_{E_k}^{\alpha/e} \right)^{-1} = 49 \text{ ms}$$

We can conclude that:

- At the beginning the electrons are more effective in the α thermalisation process. Only when $E_\alpha \leq 10 \text{ keV}$, the deuterium ions are more effective than the electrons (the tritium ions play an important role in the thermalisation process for lower energies).
- Since the energy transfer due to the collisions between α particles and electrons during the transition $E_\alpha = 3.5 \text{ MeV} \rightarrow 10 \text{ keV}$ is more important than the transition $E_\alpha = 10 \text{ keV} \rightarrow 5 \text{ keV}$, the electrons are heated more than the ions.

Exercise 3 - Runaway electrons

a) We suppose to have uniform parameters (density, temperature, ...) in the considered volume. The relation between the electric field E and the current density j is $E = \eta j$. Having $I = j \cdot A$, we can write $I = \frac{A}{\eta} E$, where $A = \pi a^2$ is the section of the cylinder. The potential difference between the extremities of the cylinder is $\Delta V_{Plasma} = E \cdot L$, therefore:

$$\Delta V_{Plasma} = \frac{\eta I}{\pi a^2} L$$

Using the numerical values, we find:

$$E \cong \frac{5.1 \times 10^{-5} Z \ln \Lambda}{(T_e [eV])^{\frac{3}{2}}} \frac{I}{\pi a^2} \cong 5.8 \times 10^{-4} \text{ V/m}$$

where the Spitzer resistivity has been used.

Finally we have:

$$\Delta V_{Plasma} = E \cdot L = 19.3 \text{ mV}$$

If we consider a stainless steel cylinder with resistivity $\eta_{inox} \approx 7 \times 10^{-7} \Omega \cdot \text{m}$, we find:

$$E_{inox} = \frac{I \eta_{inox}}{\pi a^2} = 7 \times 10^{-7} \frac{15 \times 10^6}{\pi 2^2} \text{ V/m} \approx 0.835 \text{ V/m} \Rightarrow \Delta V_{inox} = 27.8 \text{ V} \gg \Delta V_{Plasma}$$

- b) What is the behaviour of the electrons in the “tail” of the distribution function (suprathermal electrons)?

For suprathermal electrons, $v \gg v_{\text{th},e}$, we cannot use the collision frequency averaged over the Maxwellian distribution. Collisions with both ions and bulk electrons contribute to momentum loss. The collision frequency specified in the problem is

$$\nu_{se}(v) = (2 + Z) \frac{n_e e^4 \ln \Lambda}{4\pi \epsilon_0^2 m_e^2 v^3} \propto v^{-3}.$$

Taking $v > 0$ as the speed of an electron moving opposite to the electric field, its momentum equation is

$$m_e \frac{dv}{dt} = e|E| - \nu_{se}(v)m_e v = e|E| - F_c(v),$$

where $F_c(v)$ denotes the positive magnitude of the collisional friction force:

$$F_c(v) = (2 + Z) \frac{n_e e^4 \ln \Lambda}{4\pi \epsilon_0^2 m_e v^2}.$$

If $F_c(v) > e|E|$, the electron decelerates towards the thermal population. If $e|E| > F_c(v)$, the electron accelerates. As its speed increases, the friction force decreases, $F_c(v) \propto v^{-2}$, so the net accelerating force increases. This is the runaway regime within the given nonrelativistic model.

The threshold is obtained by balancing the electric and collisional forces:

$$\begin{aligned} e|E| &= F_c(v_{\text{crit}}) = (2 + Z) \frac{n_e e^4 \ln \Lambda}{4\pi \epsilon_0^2 m_e v_{\text{crit}}^2}, \\ |E| &= (2 + Z) \frac{n_e e^3 \ln \Lambda}{4\pi \epsilon_0^2 m_e v_{\text{crit}}^2}. \end{aligned}$$

Therefore, the critical kinetic energy is

$$E_{k,\text{crit}} = \frac{m_e v_{\text{crit}}^2}{2} = (2 + Z) \frac{n_e e^3 \ln \Lambda}{8\pi \epsilon_0^2 |E|}.$$

Electrons are accelerated when

$$\frac{1}{2} m_e v^2 > E_{k,\text{crit}}.$$

If T_e is expressed in eV, the corresponding thermal energy in joules is eT_e . Thus,

$$\frac{E_{k,\text{crit}}}{eT_e} = (2 + Z) \frac{n_e e^2 \ln \Lambda}{8\pi \epsilon_0^2 |E| T_e}.$$

Defining the characteristic electric field of this model as

$$E_{\text{crit}} \equiv (2 + Z) \frac{n_e e^2 \ln \Lambda}{8\pi \epsilon_0^2 T_e},$$

we obtain

$$\frac{E_{k,\text{crit}}}{eT_e} = \frac{E_{\text{crit}}}{|E|},$$

and hence the runaway condition

$$\frac{1}{2} m_e v^2 > eT_e \frac{E_{\text{crit}}}{|E|}.$$

This characteristic field has the Dreicer scaling $n_e \ln \Lambda / T_e$, with the numerical prefactor determined by the collision frequency given in the problem.

For the ITER example, using the assumed electric field,

$$|E| = \frac{|V_{\text{loop}}|}{2\pi R} = 0.58 \times 10^{-3} \text{ V m}^{-1},$$

together with $Z = 1$, $n_e = 10^{20} \text{ m}^{-3}$ and $\ln \Lambda = 17$, the formal critical kinetic energy is

$$\begin{aligned} E_{k,\text{crit}} &= (2 + 1) \frac{(1.6 \times 10^{-19})^3}{8\pi (8.85 \times 10^{-12})^2} \frac{10^{20} \times 17}{0.58 \times 10^{-3}} \\ &\simeq 1.83 \times 10^{-11} \text{ J} \simeq 114 \text{ MeV}. \end{aligned}$$

This result is a formal extrapolation of the nonrelativistic model. Since it greatly exceeds the electron rest energy, $m_e c^2 \simeq 0.511 \text{ MeV}$, a relativistic treatment is required for a physically valid runaway threshold.

Be careful with the notation: E_{crit} is a characteristic electric field, whereas $E_{k,\text{crit}}$ is a critical kinetic energy.

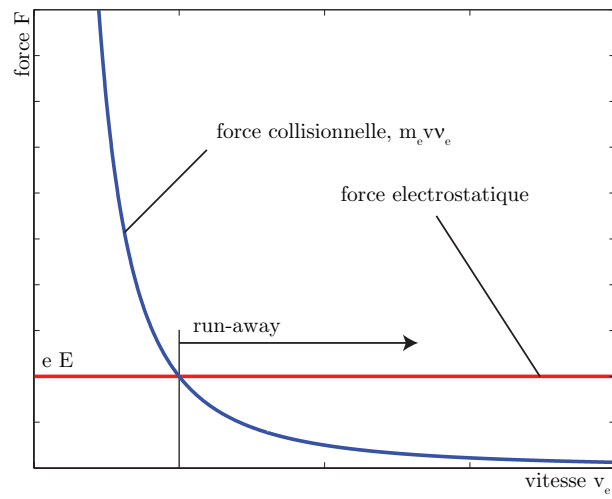


Figure 1: Electric-force magnitude $e|E|$ and collisional friction-force magnitude $F_c(v_e)$ as functions of electron speed in the nonrelativistic model.