Solution 4
Quantum Information Processing

Exercise 1 *Properties of Pauli matrices*

a) We have :

$$A = a_0 I + a_1\sigma_x + a_2\sigma_y + a_3\sigma_z = \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix}$$

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. One must have

$$\begin{cases} a_0 + a_3 = a_{11} \\ a_0 - a_3 = a_{22} \end{cases}$$

which implies $a_0 = \dfrac{a_{11} + a_{22}}{2}$ et $a_3 = \dfrac{a_{11} - a_{22}}{2}$. On the other hand one must have :

$$\begin{cases} a_1 - ia_2 = a_{12} \\ a_1 + ia_2 = a_{21} \end{cases}$$

which implies $a_1 = \dfrac{a_{12} + a_{21}}{2}$ et $a_2 = \dfrac{a_{21} - a_{12}}{2i}$.
Thus 2×2 matrices A can be written as :

$$A = \frac{a_{11} + a_{22}}{2} I + \frac{a_{12} + a_{21}}{2}\sigma_x + \frac{a_{21} - a_{12}}{2i}\sigma_y + \frac{a_{11} - a_{22}}{2}\sigma_z.$$

Note that if $A = A^\dagger$, since $\sigma_x = \sigma_x^\dagger$, $\sigma_y = \sigma_y^\dagger$, $\sigma_z = \sigma_z^\dagger$, one must also have a_0, a_1, a_2, $a_3 \in \mathbb{R}$.

b) These relations are checked by explicit calculation. Note that they are related by cyclic permutations of xyz.

c) Idem

d) <u>Diagonalization of σ_x.</u>

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Longrightarrow \begin{cases} v_1 = \lambda v_2 \\ v_2 = \lambda v_1 \end{cases}$$

$\Rightarrow v_1 = \lambda^2 v_1$ and $v_2 = \lambda^2 v_2$. To have $v_1, v_2 \neq 0$ it must be that $\lambda^2 = +1$ and thus $\lambda = \pm 1$.
The eigenvalues are ± 1.
The eigenvector corresponding to $\lambda = +1$ satisfies

$$v_1 = v_2 \qquad \text{et} \qquad v_2 = v_1.$$

$$\Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{is a normalized eigenvector}$$

The eigenvector associated to $\lambda = -1$ satisfies :

$$v_1 = -v_2 \qquad \text{et} \qquad v_2 = -v_1.$$

$$\Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \quad \text{is a normalized eigenvector}$$

Diagonalization of σ_y.

We proceed as above :

$$\det\begin{pmatrix}-\lambda & -i\\ i & -\lambda\end{pmatrix} = \lambda^2 - (-i)(i) = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

The eigenvector associated to $\lambda = +1$ satisfies

$$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = +1\begin{pmatrix}v_1\\v_2\end{pmatrix} \Rightarrow \begin{cases} -iv_2 = v_1\\ iv_1 = v_2\end{cases}$$

One can choose $v_1 = 1$ et $v_2 = i$

$$\Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \quad \text{is a normalized eigenvector.}$$

For the eigenvector associated to -1 we have :

$$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = -1\begin{pmatrix}v_1\\v_2\end{pmatrix} \Rightarrow \begin{cases} -iv_2 = -v_1\\ iv_1 = -v_2\end{cases}$$

We choose $v_1 = 1$ et $v_2 = -i$

$$\text{Thus} \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} \quad \text{is a normalized eigenvector.}$$

Diagonalization of σ_z.

$$\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad \text{in the basis} \quad \begin{pmatrix}1\\0\end{pmatrix} \quad \text{et} \quad \begin{pmatrix}0\\1\end{pmatrix}.$$

Thus 1 is the eigenvalue associated to $\begin{pmatrix}1\\0\end{pmatrix}$ and -1 the eigenvalue associated to $\begin{pmatrix}0\\1\end{pmatrix}$.

Trace. The trace of a matrix is the sum of diagonal elements and is invariant under change of basis. It is also the sum of eigenvalues. One can check that every thing is consistent $\mathrm{Tr}\,\sigma_x = \mathrm{Tr}\,\sigma_y = \mathrm{Tr}\,\sigma_z = 0$.

Determinant. The determinant equals $a_{11}a_{22} - a_{12}a_{21}$ and is also invariant under change of basis. It is also the product of eigenvalues. One can again check that everything is consistent $\det\sigma_x = \det\sigma_y = \det\sigma_z = -1$.

e) In Dirac notation a 2×2 matrix becomes :

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} \left|\uparrow\right\rangle \left\langle\uparrow\right| + a_{12} \left|\uparrow\right\rangle \left\langle\downarrow\right| + a_{21} \left|\downarrow\right\rangle \left\langle\uparrow\right| + a_{22} \left|\downarrow\right\rangle \left\langle\downarrow\right| .$$

We also remark the important relations :

$$\left\langle\uparrow\right| A \left|\uparrow\right\rangle = a_{11};\ \left\langle\uparrow\right| A \left|\downarrow\right\rangle = a_{12};\ \left\langle\downarrow\right| A \left|\uparrow\right\rangle = a_{21} \ \text{ et } \ \left\langle\downarrow\right| A \left|\downarrow\right\rangle = a_{22}.$$

We note

$$\left|\uparrow\right\rangle \left\langle\uparrow\right| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};\ \left|\uparrow\right\rangle \left\langle\downarrow\right| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};\ \left|\downarrow\right\rangle \left\langle\uparrow\right| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ \text{ et } \ \left|\downarrow\right\rangle \left\langle\downarrow\right| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

From which the expressions of Pauli matrices in Dirac notation follow.

Exercise 2 *Exponentials of Pauli matrices*

1.

$$\vec{n} \cdot \vec{\sigma} = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z \qquad \text{by definition}$$

$$\begin{aligned} (\vec{n} \cdot \vec{\sigma})^2 &= (n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z) \\ &= n_x^2\sigma_x^2 + n_y^2\sigma_y^2 + n_z^2\sigma_z^2 \\ &+ n_x n_y \sigma_x \sigma_y + n_y n_x \sigma_y \sigma_x \\ &+ n_x n_z \sigma_x \sigma_z + n_z n_x \sigma_z \sigma_x \\ &+ n_y n_z \sigma_y \sigma_z + n_z n_y \sigma_z \sigma_y \\ &= \left(n_x^2 + n_y^2 + n_z^2\right) I = I \end{aligned}$$

In the second equality we were careful to note that Pauli matrices do not commute.. In the third one we used the relation in point 2. In the last one we used that $\vec{n}$ is a unit norm vector.
This identity implies $(\vec{n} \cdot \vec{\sigma})^3 = \vec{n} \cdot \vec{\sigma}; (\vec{n} \cdot \vec{\sigma})^4 = I;$ etc...
Thus

$$\begin{aligned} \exp(it\vec{n} \cdot \vec{\sigma}) &= \sum_{k=0}^{+\infty} \frac{(it)^k}{k!} (\vec{n} \cdot \vec{\sigma})^k \\ &= \sum_{k \text{ even}} \frac{(it)^k}{k!} I + \left\{ \sum_{k \text{ odd}} \frac{(it)^k}{k!} \right\} (\vec{n} \cdot \vec{\sigma}) \end{aligned}$$

Moreover

$$\cos t = \sum_{k \text{ even}} \frac{(it)^k}{k!} \qquad \text{et} \qquad i \sin t = \sum_{k \text{ odd}} \frac{(it)^k}{k!} \tag{*}$$

(*Parenthesis :* note that

$$e^{it} = \cos t + i \sin t \qquad \text{thus :}$$

$$\sum_{k \text{ pairs}} \frac{(it)^k}{k!} + \sum_{k \text{ impairs}} \frac{(it)^k}{k!} = \cos t + i \sin t,$$

changing $t \to -t$ we also have

$$\sum_{k \text{ pairs}} \frac{(it)^k}{k!} - \sum_{k \text{ impairs}} \frac{(it)^k}{k!} = \cos t - i \sin t,$$

and adding or subtracting we find $(*)$.)
Finally we proved :

$$\exp(it\vec{n} \cdot \vec{\sigma}) = (\cos t)I + (i \sin t)\vec{n} \cdot \vec{\sigma}$$

Exercise 3 *Rotations on the Bloch sphere*

A general vector can be written in the form $\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|\downarrow\rangle$ in the Bloch sphere.

a) The eigenvectors for σ_z basis are $|\uparrow\rangle$ and $|\downarrow\rangle$, corresponding to $(\theta = 0, \phi = 0)$ and $(\theta = \pi, \phi = 0)$, respectively.

   The eigenvectors for σ_y basis are $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{i}{\sqrt{2}}|\downarrow\rangle$ and $\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{i}{\sqrt{2}}|\downarrow\rangle$, corresponding to $(\theta = \frac{\pi}{2}, \phi = \frac{\pi}{2})$ and $(\theta = \frac{\pi}{2}, \phi = -\frac{\pi}{2})$, respectively.

   The eigenvectors for σ_x basis are $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$ and $\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$, corresponding to $(\theta = \frac{\pi}{2}, \phi = 0)$ and $(\theta = \frac{\pi}{2}, \phi = \pi)$, respectively.

   The corresponding representation over the Bloch sphere is shown in Figure 1.

b) Using the general formula proved in homework 8 :

$$\exp\left(i\frac{\theta}{2}\vec{\sigma} \cdot \vec{n}\right) = \cos\left(\frac{\theta}{2}\right)I + i\vec{\sigma} \cdot \vec{n} \sin\left(\frac{\theta}{2}\right),$$

   we obtain

$$\begin{aligned}\exp\left(-i\frac{\alpha}{2}\sigma_x\right) &= \cos\left(\frac{\alpha}{2}\right)I - i\sigma_x(\sin\left(\frac{\alpha}{2}\right)) \\ &= \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) & -i\sin\left(\frac{\alpha}{2}\right) \\ -i\sin\left(\frac{\alpha}{2}\right) & \cos\left(\frac{\alpha}{2}\right) \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\exp\left(-i\frac{\beta}{2}\sigma_y\right) &= \cos\left(\frac{\beta}{2}\right)I - i\sigma_y(\sin\left(\frac{\beta}{2}\right)) \\ &= \begin{pmatrix} \cos\left(\frac{\beta}{2}\right) & -\sin\left(\frac{\beta}{2}\right) \\ \sin\left(\frac{\beta}{2}\right) & \cos\left(\frac{\beta}{2}\right) \end{pmatrix},\end{aligned}$$

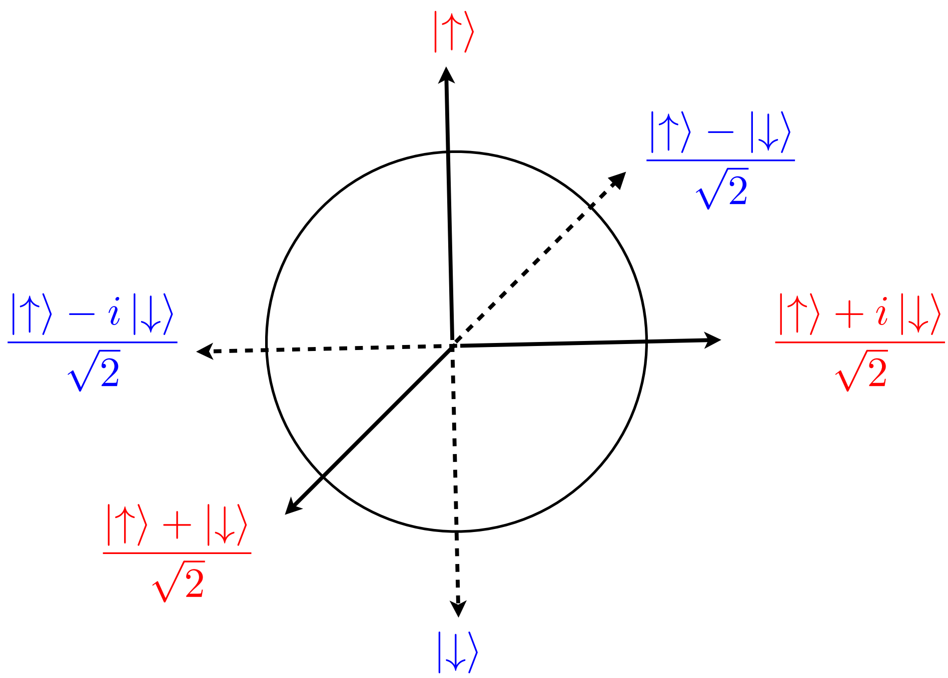

Figure 1 – Representation of basis vectors on Bloch Sphere

$$\begin{aligned}
\exp\left(-i\frac{\gamma}{2}\sigma_z\right) &= \cos\left(\frac{\gamma}{2}\right)I - i\sigma_z(\sin\left(\frac{\gamma}{2}\right)) \\
&= \begin{pmatrix} \cos\left(\frac{\gamma}{2}\right) - i\sin\left(\frac{\gamma}{2}\right) & 0 \\ 0 & \cos\left(\frac{\gamma}{2}\right) + i\sin\left(\frac{\gamma}{2}\right) \end{pmatrix} \\
&= \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}.
\end{aligned}$$

c) The matrix $\exp\left(-i\frac{\alpha}{2}\sigma_x\right)$ is a rotation matrix of angle α around the X-axis, thus the state vector $\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\frac{\pi}{2}}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle$ is transformed to the vector $\cos\left(\frac{\theta-\alpha}{2}\right)|\uparrow\rangle + e^{i\frac{\pi}{2}}\sin\left(\frac{\theta-\alpha}{2}\right)|\downarrow\rangle$. One can see the transformation geometrically on the Bloch sphere, however one can also show by direct calculation :

$$\exp\left(-i\frac{\alpha}{2}\sigma_x\right)\left(\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\frac{\pi}{2}}\sin\left(\frac{\theta}{2}\right)\right)|\downarrow\rangle = \cos\left(\frac{\theta-\alpha}{2}\right)|\uparrow\rangle + e^{i\frac{\pi}{2}}\sin\left(\frac{\theta-\alpha}{2}\right)|\downarrow\rangle.$$

Similarly, one can see that $\exp\left(i\frac{\gamma}{2}\sigma_z\right)$ is a rotation of angle γ around the Z-axis. Therefore,

$$\exp\left(-i\frac{\gamma}{2}\sigma_z\right)\left(\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\frac{\pi}{2}}\sin\left(\frac{\theta}{2}\right)\right)|\downarrow\rangle = e^{-i\frac{\gamma}{2}}\left(\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i(\frac{\pi}{2}+\gamma)}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle\right).$$