Nuclear Fusion and Plasma Physics

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Lecture 3 - 30 September 2024

Plasma state and collective effects

- Definition of a plasma
- Plasma production by ionisation
- Collective effects: Debye screening and the plasma frequency
- Models for a general description of plasmas

Single particle motion

Motion of charged particles in electric and magnetic fields

- B_0, $E = 0$ (Larmor motion, diamagnetism)
- B_0, E_0 ($E \times B$ drift)
- $B(x)$, variation across B, $E = 0$ (grad B and curvature drifts)
- $B(x)$, variation along B, $E = 0$ (mirror effect)

Suggested additional reading

a) A.Fasoli, Plasma Physics II Lecture Notes, Chapters 1 and 2, and
Appendix A (https://crppwww.epfl.ch/physplas2/repository/2011/Fasoli_PlasmasII.pdf)

b) J.Freidberg's book, Chapter 8 (8.1-8.7 and 8.9-8.10)

1 The plasma state

What is a plasma ? "An ionised, quasi-neutral gas, exhibiting collective effects"

1.1 "Ionised"

Result of the process of ionisation. In most cases of interest to us for fusion, **impact** ionisation is the dominant process (Fig. 1 & Fig. 2):

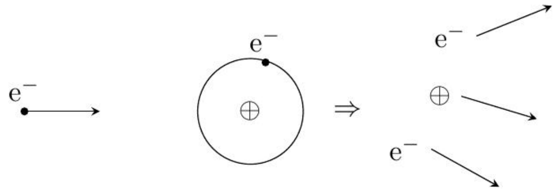


Fig. 1: Ionisation process

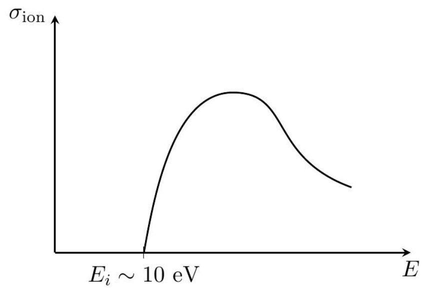


Fig. 2: Ionisation cross section

For a plasma to be in equilibrium, there has to be the same number of ionisations and **recombination events**.

Recombination: In fusions plasmas it will be dominated by "radiative" recombination, illustrated in Fig. 3 and Fig. 4.

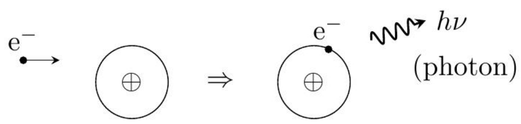


Fig. 3: Recombination process

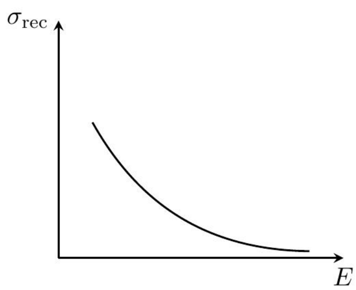


Fig. 4: Recombination cross section

The equilibrium is given by:

$$n_e < \sigma_{\rm rec} v_e > = n_n < \sigma_{\rm ion} v_e > \qquad <>= \text{ average over distribution function}$$

$\uparrow$ $\qquad\qquad$ $\uparrow$

'targets' are not the same for the two processes !

Note 3.1.1: This equation sets an equality between the frequencies at which ionisation and recombination processes occur, hence an equilibrium.

A very simple equilibrium is that of the **solar corona** (between impact ionisation and radiative recombination). A global thermodynamical equilibrium is described by the Saha equation:

$$\frac{n_e}{n_n} \approx 3 \cdot 10^{27} \frac{T^{3/2}\ [\text{eV}]}{n_i\ [\text{m}^{-3}]} \exp\{-\frac{E_i}{T}\}$$

where:

$$\begin{aligned} T_e &= T_i = T_n = T \\ E_i &= \text{ionisation energy} \end{aligned}$$

and defining $\alpha = \frac{n_e}{n_e+n_n}$ the relative degree of ionisation :

$$\alpha = \frac{1}{1 + \frac{n_i\, e^{E_i/T}}{3 \cdot 10^{27}\, T^{3/2}}}$$

This relation is presented in Fig. 5.

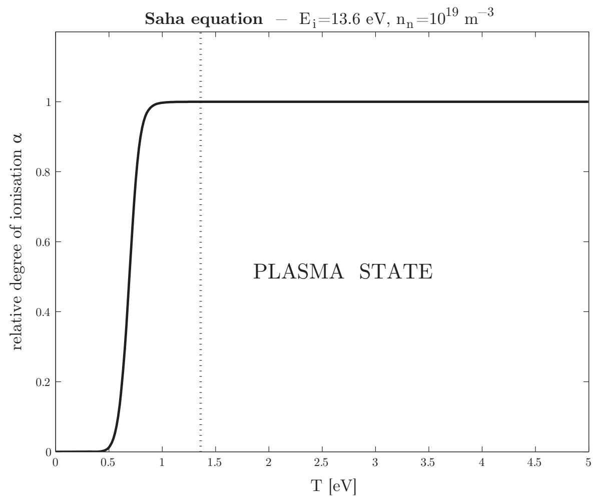


Fig. 5: Relative degree of ionisation α as a function of temperature.

The sharp transition from exponential signals a "phase transition" which defines the plasma as the 4$^{\text{th}}$ state of matter.

Various plasmas

Ex. 1 : Air ($\sim N_2$, $E_i = 14.5$ eV), $T \sim 1/40$ eV, $n_n \approx 10^{25}$ m^{-3}, $\frac{n_e}{n_n} \sim \alpha \sim 10^{-120} \sim 0$!

Ex. 2 : Solar corona $T \sim 500$ eV, $n_e \sim 10^{13}$ m^{-3} $\Rightarrow \alpha \sim 1$

1.2 "Globally neutral" ('quasi-neutrality')

A globally neutral plasma refers to a state where $n_e \approx n_i$ at least on average (both spatially and temporally). In the exercises you have demonstrated 'quantitatively' how difficult it is to violate quasi-neutrality.

Naturally, if we look at a microscopic level, this neutrality will be violated, for e.g., if we approach an individual ion. So, the question is, how close should we be to the single charge to feel its field ?

This question leads us to the 3$^{\text{rd}}$ aspect of the definition of a plasma : collective effects.

1.3 "Collective effects"

1.3.1 Static: charge (or potential) screening

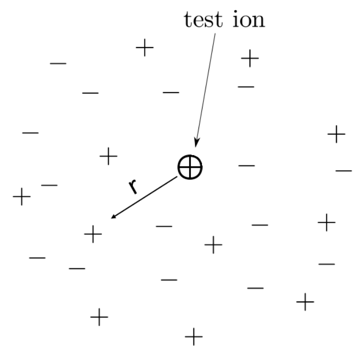


Fig. 6: An extra ion is inserted into a plasma

Insert an extra positive charge (fig. 6):

- What is the potential around the extra positive charge?
- How does the density of electrons change around it?
- Up to which distance will the perturbation caused by the extra positive charge be felt?

Solution in a simple situation:

- $T_i = 0$ (ions don't move); $n_i = n_0$, and $n_e = n_0$ for $r \to \infty$
- The electrons are distributed according to the Maxwell-Boltzmann : $n_e = n_0 \exp\{-\frac{\text{energy}}{T}\} = n_0 \exp\{\frac{e\phi(r)}{T}\}$
- The perturbation is small [quasi-neutrality !] : $\frac{e\phi}{T} \ll 1$
- Singly ionised ions ($q_i = +e$)

Starting from Maxwell's equations, we have the relation for the divergence of the electric field E:

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0} \Rightarrow \nabla^2 \phi = -\frac{\rho}{\varepsilon_0}$$

Next, we express the charge density ρ in terms of the ion and electron densities n_i and n_e:

$$\rho = e(n_i - n_e)$$

Assuming a globally neutral plasma in equilibrium, we have $n_i \approx n_0$ (where n_0 is the equilibrium density of ions), and the electron density n_e can be expressed using the Boltzmann distribution:

$$n_e = n_0 e^{e\phi/T}$$

For small perturbations in potential ϕ, we can approximate the exponential function using a Taylor series expansion around $\phi = 0$:

$$e^{e\phi/T} \approx 1 + \frac{e\phi}{T}$$

Substituting this into the expression for ρ, we get:

$$\rho = en_0 \left(1 - e^{e\phi/T}\right) \approx en_0 \left(1 - 1 - \frac{e\phi}{T}\right) = -\frac{e^2 n_0}{T} \phi$$

This shows that the charge density ρ is proportional to the potential ϕ with a proportionality constant $\frac{e^2 n_0}{T}$.

Substituting this result back into our modified Poisson equation:

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0} = \frac{e^2 n_0}{\varepsilon_0 T} \phi$$

This is a differential equation describing the behavior of the electric potential in a plasma. We can highlight this important result:

$$\boxed{\nabla^2 \phi = \frac{e^2 n_0}{\varepsilon_0 T} \phi}$$

Note 3.1.2 : $\left(\sqrt{\frac{e^2 n_0}{\varepsilon_0 T}}\right)^{-1}$ has dimensions of length. Define $\lambda_D \equiv \sqrt{\frac{\varepsilon_0 T}{e^2 n_0}}$ "Debye length".

Note 3.1.3: In spherical coordinates, there is only an r dependence (problem is spherically symmetric).

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

Substitute

$$U(r) = \phi(r)r$$

This yields

$$\begin{aligned}
\frac{\partial\phi}{\partial r} &= \frac{\partial}{\partial r}\left(\frac{1}{r}U(r)\right) = -\frac{1}{r^2}U(r) + \frac{1}{r}\frac{\partial U}{\partial r} \\
\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left[-\frac{1}{r^2}U + \frac{1}{r}\frac{\partial U}{\partial r}\right]\right\} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left\{-U + r\frac{\partial U}{\partial r}\right\} = \frac{1}{r^2}\left\{-\frac{\partial U}{\partial r} + \frac{\partial U}{\partial r} + r\frac{\partial^2 U}{\partial r^2}\right\} \\
&= \frac{1}{r}\frac{\partial^2 U}{\partial r^2} = \frac{U}{r}\frac{1}{\lambda_D^2} \Rightarrow \boxed{\frac{1}{\cancel{r}}\frac{\partial^2 U}{\partial r^2} = \frac{U}{\cancel{r}}\frac{1}{\lambda_D^2}}
\end{aligned}$$

We have defined the **Debye length**:

$$\lambda_D = \sqrt{\frac{\varepsilon_0 T}{e^2 n}} \begin{array}{ll} \leftarrow & \text{thermal motion makes shielding difficult} \\ \leftarrow & \text{space charge helps shielding} \end{array}$$

<u>Solution</u> :

$$\begin{aligned}
U(r) &= Ae^{-r/\lambda_D} + Be^{r/\lambda_D} \\
\text{or } \phi(r) &= A\frac{e^{-r/\lambda_D}}{r} + B\frac{e^{r/\lambda_D}}{r}
\end{aligned}$$

Boundary conditions :

$$\left\{\begin{array}{lll} \phi(r) \overset{r\to\infty}{\to} 0 & \Rightarrow & B = 0 \\ \underbrace{\phi(r) \overset{r\to 0}{\sim} \frac{e}{4\pi\varepsilon_0 r}}_{\text{single particle potential}} & \Rightarrow & A = \frac{e}{4\pi\varepsilon_0} \end{array}\right.$$

Density :

$$n_e(r) \cong n_0(1 + \frac{e\phi}{T}) = n_0\left\{1 + \underbrace{\frac{e^2}{4\pi\varepsilon_0 T}\frac{1}{r}e^{-r/\lambda_D}}_{>0 \text{ as expected}}\right\}$$

there are slightly more electrons around the positive charge, but this extra electron density goes to 0 as $r \to \infty$, with λ_D as the exponential decay length

$$\Rightarrow \boxed{\phi(r) = \frac{e}{4\pi\varepsilon_0}\frac{1}{r}\cdot e^{-r/\lambda_D}}$$

where e^{-r/λ_D} is the term due to plasma collective interactions.

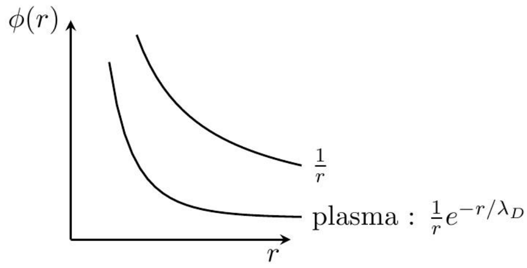


Fig. 7: Comparison of the potential $\phi(r)$ in vacuum $(\frac{1}{r})$ and in a plasma $\left(\frac{1}{r}e^{-r/\lambda_D}\right)$, showing the effect of Debye shielding which causes the potential to decay exponentially beyond the Debye length.

Note 3.1.4: In order for this effect to happen, i.e. for the plasma to exhibit collective effects, we must have:

- enough particles within the sphere with radius equal to λ_D : $N_D = \frac{4}{3}\pi\lambda_D^3 n_0 \gg 1$
- plasma size $> \lambda_D$

1.3.2 Dynamical effect - Plasma Oscillations

By combining the characteristic velocity ($v = \sqrt{\frac{T_e}{m_e}}$) and length ($\lambda_d$), we can find a characteristic time (or frequency) for the plasma response to local violation of quasi-neutrality :

$$\omega_p \sim \frac{1}{\tau} \sim \frac{v}{\lambda_D} = \sqrt{\frac{e^2 n_e}{\epsilon_0 m_e}}$$

This is the **plasma frequency**. Numerically :

$$f_p = \frac{\omega_p}{2\pi} \simeq 9000\sqrt{n\,[\mathrm{cm}^{-3}]}\ \mathrm{Hz}$$

Note 3.1.5: To propagate a wave in a simple plasma (no B-field) one should have $\omega > \omega_p$.

2 General problem of plasma description

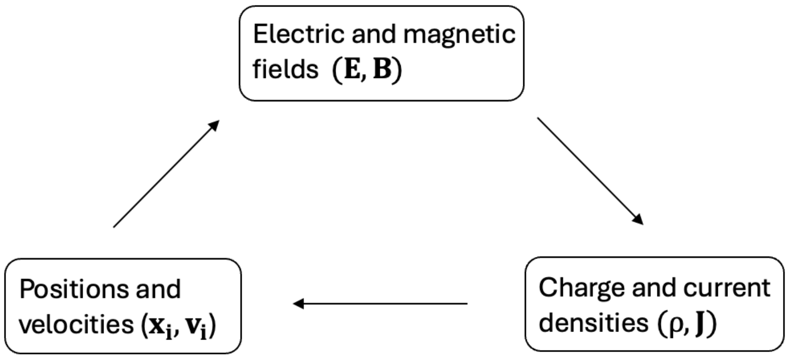


Fig. 8: "Typical self-consistent loop which appears in models of plasma physics. The positions and velocities of particles $(\mathbf{x}_i, \mathbf{v}_i)$ determine the charge and current densities $(\rho, \mathbf{J})$, which in turn generate the electric and magnetic fields $(\mathbf{E}, \mathbf{B})$. These fields influence the particles' positions and velocities, completing the feedback loop."

How to combine the three steps ? We need a model to describe the plasma.

Three models :

1. Single particle (eq. of motion) $\begin{cases} \text{orbits} \\ \text{magnetic confinement} \end{cases}$
2. Kinetic Boltzmann (Vlasov) $\begin{cases} \text{transport} \\ \text{wave-particle interactions} \end{cases}$
3. Fluid (eq. motion, continuity, eq. of state) $\begin{cases} \text{macro-stability (equilibrium)} \\ \text{waves} \end{cases}$

Today we discuss the single-particle model, in particular step A of Fig.8, i.e. how $\mathbf{E}, \mathbf{B}$ affect particle motion.

Eq. of motion (non relativistic) :

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{F}_{\text{ext}} \quad \Longrightarrow \quad \begin{cases} \mathbf{x}(t) \\ \mathbf{v}(t) \end{cases} \quad \text{for given} \begin{cases} \mathbf{E}(\mathbf{x}, t) \\ \mathbf{B}(\mathbf{x}, t) \end{cases}$$

As a plasma is made up of charged particles, it is interesting to study the behavior of such particles under various configurations of electromagnetic fields. Different important cases will be considered in the following:

1. $\mathbf{B} = \mathbf{B}_0$; $\mathbf{E} = 0$ Larmor motion (diamagnetism)
2. $\mathbf{B} = \mathbf{B}_0$; $\mathbf{E} = \mathbf{E}_0$ "$\mathbf{E} \times \mathbf{B}$ drift"
3. $\mathbf{B} = \mathbf{B}(\mathbf{x})$; $\mathbf{E} = 0$ "∇B" and curvature drift
   with $\nabla B \perp \mathbf{B}$ (variation across $\mathbf{B}$)

4. $\mathbf{B} = \mathbf{B}(\mathbf{x})\,;\ \mathbf{E} = 0$
   with $\nabla B \parallel \mathbf{B}$ (variation along $\mathbf{B}$) $\rightarrow$ magnetic mirror

Naturally, there exist other, more complicated situations, but the main aspects are captured in the four simple cases illustrated here.

2.1 $\mathbf{B}_0$ uniform, no $\mathbf{E}$

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\mathbf{v} \times \mathbf{B}_0$$

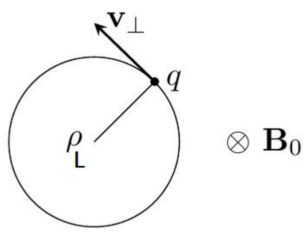


Fig. 9: Circular motion

$$\frac{mv_\perp^2}{\rho_L} = m\rho_L\Omega^2 = |q|\,v_\perp B_0 \quad \rightarrow \quad \begin{cases} \rho_L &= \dfrac{v_\perp}{\Omega} \\ \Omega &= \dfrac{|q|\,B_0}{m} \end{cases} \quad \text{(from laws of circular motion, } v_\perp = \rho\Omega\text{)}$$

Numerical values :

$$f = \frac{\Omega}{2\pi} = \begin{cases} \text{el.} & 28 \times B[\mathrm{T}]\ [\mathrm{GHz}] & \text{ex : ITER } f_e \sim 170 \text{ GHz} \\ \text{ions} & 15.2\frac{Z}{A} \times B[\mathrm{T}]\ [\mathrm{MHz}] & \text{ex : ITER } f_i \sim 50 \text{ MHz} \end{cases}$$

$$\rho_L = \frac{v_\perp}{\Omega} = \begin{cases} \text{el.} & 10^{-4}\frac{\sqrt{T_e\ [\mathrm{keV}]}}{B_0\ [\mathrm{T}]}\ [\mathrm{m}] & (\text{ex : 10 keV, B=3T} \rightarrow \rho_{L_e} \simeq 0.1\,\mathrm{mm}) \\ \text{ions} & 5 \cdot 10^{-3}\frac{\sqrt{T_e\ [\mathrm{keV}]}}{B_0\ [\mathrm{T}]}\sqrt{\frac{m_i}{m_p}}\ [\mathrm{m}] & (\text{ex : 10 keV, B=3T, H-ions} \rightarrow \rho_{L_i} \simeq 7\,\mathrm{mm}) \end{cases}$$

Note 3.2.1: from our “design of a reactor” study we know that $B \sim 5\,\mathrm{T}$, plasma radius $>> 7\,\mathrm{mm}$ so these values are ok.

Note 3.2.2:

- Energy is conserved (force is $\perp$ to $\mathbf{v}$: no work !) NB : **we neglect losses by radiation**
- Parallel and perpendicular motion are decoupled (with respect to $\mathbf{B}_0$)
- In perpendicular plane, the trajectory of the particle is a circle of radius ρ_L (Larmor radius): this is a cyclotron motion, with frequency Ω

Note 3.2.3: sign of gyro-motion gives reduction of $\mathbf{B}_0$: **diagmagnetism** [remember : this is why $\beta = \frac{nT}{B^2/2\mu_0} < 1$]

2.2 $\mathbf{B}_0$ and $\mathbf{E}_0$ uniform

As we did before, we project the movement along parallel and perpendicular directions with respect to the magnetic field:

$$\parallel \text{ motion : } m\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = qE_\parallel \qquad \left\{ \begin{array}{lcll} v_\parallel(t) & = & v_{\parallel 0} + \frac{q}{m}E_0 t & \text{(nothing special)} \\ z(t) & = & z_0 + v_{\parallel 0}t + \frac{1}{2}\frac{q}{m}E_0 t^2 & \end{array} \right.$$

$\perp$ motion : sketch for $\mathbf{E}_0 \perp \mathbf{B}_0$:

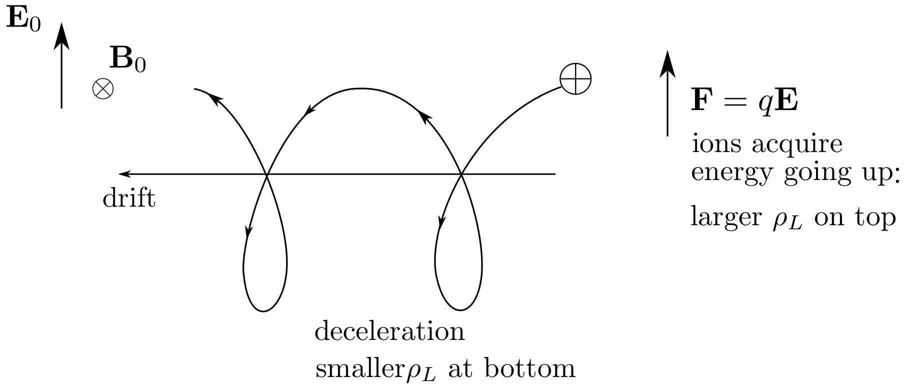


Fig. 10: Motion of ions in a constant magnetic field $\mathbf{B}_0$ and a perpendicular electric field $\mathbf{E}_0$. The ions have a helical motion due to the Lorentz force, which results in a drift perpendicular to both fields. As shown in the diagram, the ions accelerate upwards when moving with the electric field, resulting in a larger Larmor radius ρ_L at the top of the trajectory and decelerate when moving against the electric field, resulting in a smaller ρ_L at the bottom. This creates a net drift motion.

And the same holds for electrons

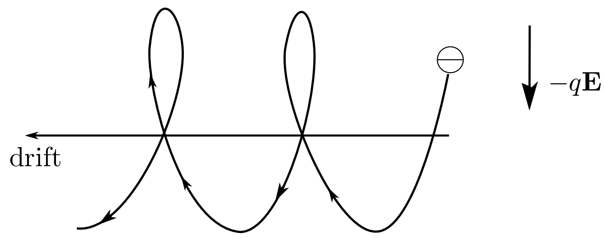


Fig. 11: Motion of electrons in a constant magnetic field $\mathbf{B}_0$ and a perpendicular electric field $\mathbf{E}_0$. The electrons experience a helical motion due to the Lorentz force, resulting in a drift perpendicular to both fields. The electrons accelerate in the direction opposite to the electric field $-\mathbf{E}_0$, resulting in a larger Larmor radius ρ_L at the bottom of the trajectory and decelerate when moving in the same direction as the electric field, resulting in a smaller ρ_L at the top. This creates a net drift motion similar to the ions but in the opposite direction due to the opposite charge.

$\Rightarrow$ **Drifts to the left for both species.**

How to calculate this steady-state drift ?

<u>Definition</u>: Let $\mathbf{v}_L$ denote the solution of case $\mathbf{B} = \mathbf{B}_0$; $\mathbf{E} = 0$ (Larmor motion)

We look for $\mathbf{v}_d$, such that $\mathbf{E}_0 + \mathbf{v}_d \times \mathbf{B}_0 = 0$; in fact $\mathbf{v}_L + \mathbf{v}_d$ is the solution ($\mathbf{v}_d$ = constant).

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q(\mathbf{E}_0 + \mathbf{v} \times \mathbf{B}_0);\ \text{but } \mathbf{v}_L : m\frac{\mathrm{d}\mathbf{v}_L}{\mathrm{d}t} = q\mathbf{v}_L \times \mathbf{B}_0$$

$$m\frac{\mathrm{d}\mathbf{v}_L}{\mathrm{d}t} + m\frac{\mathrm{d}\mathbf{v}_d}{\mathrm{d}t} = q\mathbf{E}_0 + q\mathbf{v}_L \times \mathbf{B}_0 + q\mathbf{v}_d \times \mathbf{B}_0$$

So, we need to solve[1] $\boxed{\mathbf{E}_0 + \mathbf{v}_d \times \mathbf{B}_0 = 0}$ $\quad (\times \mathbf{B}_0)$

$$\mathbf{E}_0 \times \mathbf{B}_0 + (\mathbf{v}_d \times \mathbf{B}_0) \times \mathbf{B}_0 = 0 \quad \Rightarrow \quad \mathbf{E}_0 \times \mathbf{B}_0 + \underbrace{(\mathbf{v}_d \cdot \mathbf{B}_0)\mathbf{B}_0}_{\parallel \text{ to } B_0} - \mathbf{v}_d B_0^2 = 0$$

$\perp$ motion (the one we are interested in) :

$$\mathbf{E}_0 \times \mathbf{B}_0 = \mathbf{v}_{d_\perp} B_0^2 \quad \Rightarrow \quad \boxed{\mathbf{v}_{d_\perp} = \mathbf{v}_{E\times B} = \frac{\mathbf{E}_0 \times \mathbf{B}_0}{B_0^2}} \tag{2.1}$$

$\rightarrow$ "E cross B drift" (of "guiding center") as expected from sketch. The overall motion in this configuration is hence given by a Larmor motion but with a shift caused by the drift velocity induced by the electric field.

[1] $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \times \mathbf{b}) \times \mathbf{a} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Note 3.2.4:

- $\mathbf{v}_d$ is independent of charge, m, v.
- same for ions, electrons $\rightarrow$ no current, no charge separation

Generalisation of $\mathbf{v}_{E\times B}$

Instead of $q\mathbf{E}_0$, we can put any force $\mathbf{F}_{\text{ext}} = q\mathbf{E}_0$, or $\mathbf{E}_0 = \frac{\mathrm{F}_{\text{ext}}}{q}$

$$\Rightarrow \mathbf{v}_F = \frac{\mathbf{F}_{\text{ext}}/q \times \mathbf{B}_0}{B_0^2} \qquad \text{ex : gravity : } \mathbf{v}_g = \frac{m\mathbf{g}_{\text{ext}}/q \times \mathbf{B}_0}{B_0^2} \tag{2.2}$$

Note 3.2.5: If $\mathbf{F}_{\text{ext}}$ does **not** contain the charge, then the drift **does** depend on the charge; this leads to charge separation.

2.3 $\mathbf{B}(\mathbf{x})$ and $\mathbf{E} = 0$, with $\nabla B \perp \mathbf{B}$ (variation across $\mathbf{B}$)

Consider two limiting cases :

(3a) $|\mathbf{B}|$ changes, but **not** the direction

(3b) only the direction of $\mathbf{B}$ changes ($\mathbf{B}/|\mathbf{B}|$ changes along $\mathbf{B}$)

(3a)

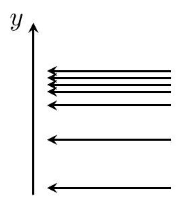


Fig. 12: $|\mathbf{B}|$ increases with y (vertically)

The calculation is complicated, but if the variations are not too fast (in space and in time), we can always decompose the motion into $\begin{cases} \text{Larmor motion} \\ \text{drift of the guiding center} \end{cases}$

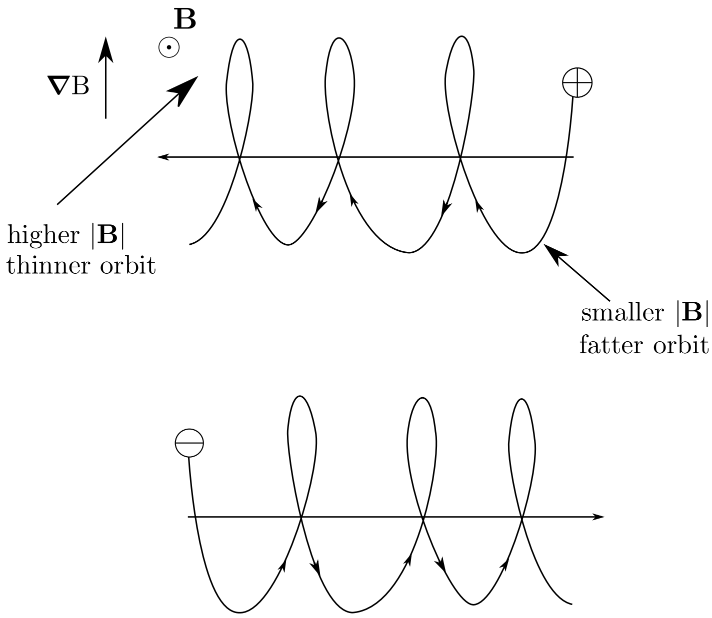
Conditions :

$$L_\perp = \left|\frac{B}{\nabla_\perp B}\right| \gg \rho_L \quad \text{the variation occurs over distances larger than } \rho_L$$

$$L_\parallel = \left|\frac{B}{\nabla_\parallel B}\right| \gg v_\parallel \underbrace{\frac{2\pi}{\Omega}}_{\text{space covered in one Larmor orbit}}$$

$$\text{time} \qquad \left|\frac{1}{B}\frac{\partial B}{\partial t}\right| \ll \Omega$$

Sketch :



Drift should be to the
$\begin{cases} \text{left} & \text{for ions} \\ \text{right} & \text{for electrons} \end{cases}$

$$\mathbf{v} \sim q\mathbf{B} \times \nabla B$$

Fig. 13: Motion of ions (upper) and electrons (lower).

Precise calculation gives in fact

$$\boxed{\mathbf{v}_d = \frac{1}{2q} m \mathbf{v}_\perp^2 \frac{\mathbf{B} \times \nabla B}{B^3}} = \mathbf{v}_{\nabla B} \qquad \text{“grad B drift”}$$

Note 3.2.6: The drift velocity $\mathbf{v}_{\nabla B}$ depends on q, and thus induces:

⇒ current

⇒ charge separation

(3b) Variation in direction of **B**

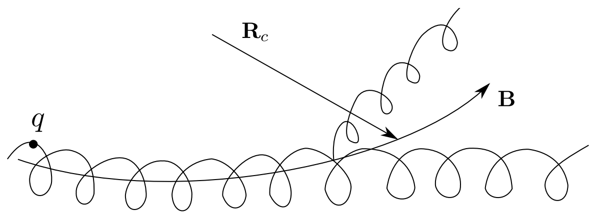


Fig. 14: Sketch of the trajectory of two charges of different signs in a region of magnetic field curvature

The guiding center 'feels' a centrifugal force. Noting R_c the radius of curvature, we have :

$$\mathbf{F}_{\text{gc}} = m \frac{v_\parallel^2}{R_c} \frac{\mathbf{R}_c}{R_c} \equiv \mathbf{F}_{\text{ext}}$$

Recalling the general formula for the drift velocity generated by an external force $\mathbf{F}_{\text{ext}}$ (Eq. 2.2), we can write the “curvature drift” as :

$$\mathbf{v}_{\text{curv}} = \frac{1}{q B^2} m \frac{v_\parallel^2}{R_c} \frac{\mathbf{R}_c}{R_c} \times \mathbf{B} = \frac{1}{q} \frac{m v_\parallel^2}{R_c^2} \frac{\mathbf{R}_c \times \mathbf{B}}{B^2}$$

<u>Case of vacuum fields</u> (how to confine a plasma ?)

- Combine $\mathbf{v}_{\nabla B}$ and $\mathbf{v}_{\text{curv}}$
- In vacuum (in cyl. coordinates) $|\mathbf{B}| \propto \dfrac{1}{R_c}$ so that
$$\frac{\nabla B}{B} = -\frac{1}{R_c^2}\mathbf{R}_c$$

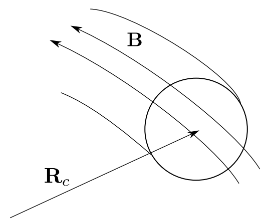


Fig. 15: Confining a plasma

So

$$\begin{aligned}
\mathbf{v}_d^{\text{total}} &= \frac{1}{2q}mv_\perp^2\frac{\mathbf{B}\times\nabla B}{B^3} + \frac{1}{q}mv_\parallel^2\frac{\mathbf{R}_c\times\mathbf{B}}{B^2R_c^2} \\
&= \frac{1}{2q}mv_\perp^2\frac{\mathbf{R}_c\times\mathbf{B}}{B^2R_c^2} + \frac{1}{q}mv_\parallel^2\frac{\mathbf{R}_c\times\mathbf{B}}{B^2R_c^2} \\
&= \left[\frac{1}{2}mv_\perp^2 + mv_\parallel^2\right]\frac{1}{q}\frac{\mathbf{R}_c\times\mathbf{B}}{B^2R_c^2} \\
&= \text{if we average energy over distribution :} \\
&\quad <\frac{1}{2}mv_\perp^2>= T \qquad \text{(2 degrees of freedom)} \\
&\quad <\frac{1}{2}mv_\parallel^2>= \frac{T}{2} \qquad \text{(1 degree of freedom)}
\end{aligned}$$

$$\Rightarrow \boxed{<\mathbf{v}_d^{\text{total}}>= \frac{2T}{q}\frac{\mathbf{R}_c\times\mathbf{B}}{B^2R_c^2}}$$

<u>**Note 3.2.7**</u>:

- $\mathbf{v}_{\text{curv}}$ and $\mathbf{v}_{\nabla B}$ add up (unfortunately !)
- dependence on $q \to$ charge separation: ions and electrons drift in opposite directions
- Proportional to energy, no mass dependence

Magnitude :

$$\left|<\mathbf{v}_d^{\text{total}}>\right| = \frac{2T}{|q|}\frac{1}{BR_c} = v\frac{\rho_L}{R_c}$$

<u>Can we conclude anything about particle confinement ?</u>

Simplest idea : particles move freely along $\mathbf{B}$ but **not** across : let's confine them in a **toroidal** system (no beginning, no end).

We need a more complicated $\mathbf{B}$-field structure to confine particles ! ($\to$ rotational transform, tokamak, . . .)

What happens ?
[This is done in the exercices]

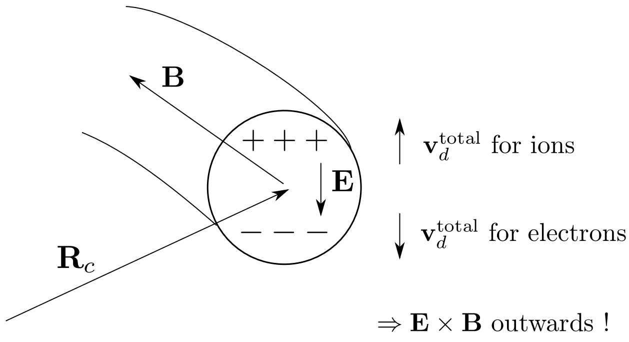


Fig. 16: $\mathbf{E}\times\mathbf{B}$ outwards $\Rightarrow$ **no confinement**

2.4 $\mathbf{B}(\mathbf{x})$ and $\mathbf{E}=0$, with $\nabla B \parallel \mathbf{B}$ (variation of $|\mathbf{B}|$ along $\mathbf{B}$)

$\nabla B \parallel \mathbf{B}$

$L_\parallel = \left|\frac{B}{\nabla_\parallel B}\right| \gg v_\parallel \underbrace{\frac{2\pi}{\Omega}}_{\text{space covered in one Larmor orbit}}$ (**to have Larmor motion**)

Cylindrical coordinates (r,θ,z) :

- $B_\theta = 0 \rightarrow B_z = B_z(z)$
- but $B_r \neq 0$ (field lines are not exactly along z)

$$\nabla\cdot\mathbf{B}=0 \quad \overset{\text{cyl. coord.}}{\Rightarrow} \quad \frac{1}{r}\frac{\partial}{\partial r}(rB_r)+\frac{\partial B_z}{\partial z}=0 \quad \rightarrow \frac{\partial}{\partial r}(rB_r) = -r\frac{\partial B_z}{\partial z}$$

$$rB_r = -\int_0^r \mathrm{d}r'\left(r'\frac{\partial B_z}{\partial z}\right) \quad \overset{\text{small distances}}{\simeq} \quad \left(-\left.\frac{\partial B_z}{\partial z}\right|_{r=0}\right)\frac{r^2}{2} \quad \Rightarrow B_r \simeq -\frac{r}{2}\left.\frac{\partial B_z}{\partial z}\right|_{r=0}$$

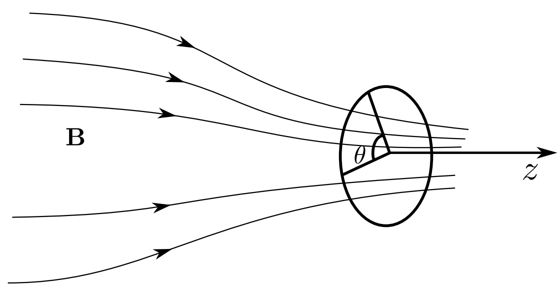


Fig. 17: Description with cylindrical coordinates

<u>Note 3.2.8</u>: $|B| = |B(r,z)| \quad \rightarrow \quad \mathbf{B}\times\nabla B$ drift is azimuthal (not important).

We are interested in **parallel** motion.

Consider a particle whose guiding center lies on the $z=0$ axis : $r \sim \rho_L$

$\boxed{B_r \simeq -\frac{\rho_L}{2}\frac{\partial B_z}{\partial z}}$ {"little" B_r produced by the fact that B varies with z}

which parallel force does this produce ?

Parallel force (with $B_\theta = 0$ and $v_\theta = \mp v_\perp = -\frac{|q|}{q}v_\perp$):

$$\begin{aligned}
q(\mathbf{v}\times\mathbf{B})_z &= q(v_r B_\theta - v_\theta B_r) = -q(\mp v_\perp)B_r = +q\frac{|q|}{q}v_\perp B_r \\
&= |q|\, v_\perp B_r \simeq |q|\, v_\perp \left(-\frac{\rho_L}{2}\frac{\partial B_z}{\partial z}\right) = -\,|q|\, v_\perp \underbrace{\frac{v_\perp m}{|q|\,B}}_{\rho_L}\frac{1}{2}\frac{\partial B_z}{\partial z} \\
&= -\underbrace{\frac{1}{2}\frac{v_\perp^2 m}{B}}_{\mu \text{ ``\textbf{magnetic moment}'' (corresponds to current}\times\text{area)}}\frac{\partial B_z}{\partial z} = -\mu\frac{\partial B_z}{\partial z}
\end{aligned}$$

So, both electrons and ions feel a force that tries to prevent them from moving towards **increasing** field.

Eq. of motion ($\parallel$ to $\mathbf{B}$) :

$$\boxed{m\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = -\mu\nabla_\parallel B}$$

This is a very useful form, as, in the conditions of interest here, μ **is conserved** ["**adiabatic invariant**", i.e. constant if the changes in the system are slow (in time and space)].

μ is constant along the particles motion : as $\mu = \frac{1/2mv_\perp^2}{B}$, if B increases, $v_\perp$ must increase.

But the total kinetic energy is constant, therefore $v_\perp^2 + v_\parallel^2 = \text{const}$, thus if $v_\perp$ increases, $v_\parallel$ must decrease. This leads to an interesting situtation for confinement: the 'magnetic mirror'.

Magnetic mirror

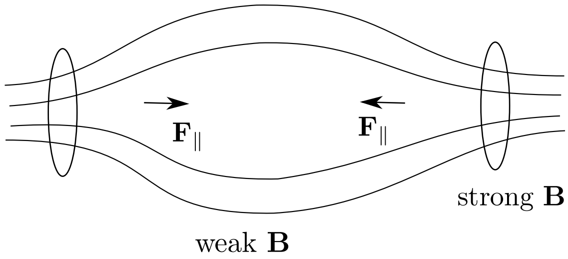


Fig. 18: Magnetic mirror

$$\frac{mv_{\parallel}^2}{2} = \text{ Energy} - \mu B$$

If $\mathbf{B}$ is large enough in the throat, $v_{\parallel} \to 0$ and can change sign [both for ions and electrons] The particles are hence "reflected" when they reach this region with large magnetic field. This results in a confinement in the direction parallel to $\mathbf{B}$.

Note 3.2.9: This is how particles are confined in Van-Allen belts in the Earths dipole field.

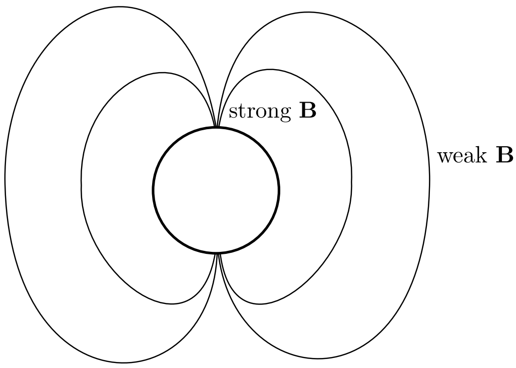


Fig. 19: Earth dipole field

[ions from thermonuclear explosions stay confined for many years !]

Note 3.2.10 : Magnetic mirrors were the first confinement schemes to be tested for fusion. But they have a **problem** : not all particles are confined! E.g. : if $v_{\perp} = 0 \to \mu = 0 \to F_{\parallel} = 0$. One needs a large enough ratio $v_{\perp}/v_{\parallel}$.

In the exercises today you'll evaluate this limit ("loss cone").