Nuclear Fusion and Plasma Physics

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Lecture 3 - 09 October 2023

Plasma state and collective effects

- Definition of plasma
- Plasma production by ionisation
- Collective effects: Debye screening and the plasma frequency
- Models for a general description of plasmas

Single particle motion

Motion of charged particles in electric and magnetic fields

- B_0, $E = 0$ (Larmor motion, diamagnetism)
- B_0, E_0 ($E \times B$ drift)
- $B(x)$, variation across B, $E = 0$ (grad B and curvature drifts)
- $B(x)$, variation along B, $E = 0$ (mirror effect)

Suggested additional reading

A.Fasoli, Plasma Physics II Lecture Notes, Chapters 1 and 2, and
Appendix A (https://crppwww.epfl.ch/physplas2/repository/2011/Fasoli_PlasmasII.pdf)

J.Freidberg's book, Chapter 8 (8.1-8.7 and 8.9-8.10)

1 The plasma state

What is a plasma ? "An ionised gas, quasi-neutral, exhibiting collective effects"

1.1 "Ionised"

Result of the process of ionisation. In most cases of interest for us, for fusion, **impact** ionisation is the dominant process :

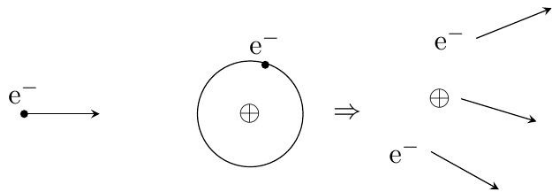


Fig. 1: Ionisation process

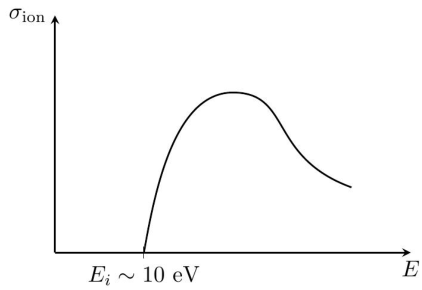


Fig. 2: Ionisation cross section

For a plasma to be in equilibrium, there has to be the same number of ionisations and **recombination events**.

Recombination : in our plasmas it'll be dominated by "radiative" recombination

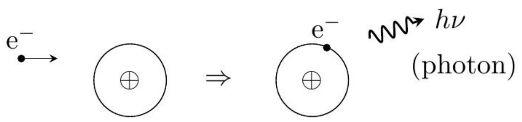


Fig. 3: Recombination process

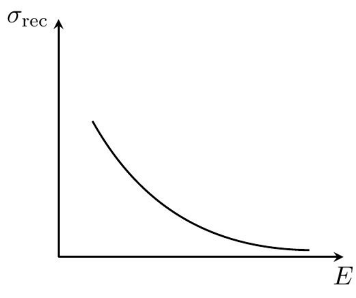


Fig. 4: Recombination cross section

The equilibrium is given by

$$n_e < \sigma_{\rm rec} v_e > = n_n < \sigma_{\rm ion} v_e > \qquad <> = \text{ average over distribution function}$$

$\uparrow$ $\qquad\qquad$ $\uparrow$

'targets' are not the same for the two processes !

A very simple equilibrium is that of the **solar corona** (between impact ionisation and radiative recombination), for a global thermodynamical equilibrium → Saha equation

$$\frac{n_e}{n_n} \approx 3 \cdot 10^{27} \frac{T^{3/2}\ [\text{eV}]}{n_i\ [\text{m}^{-3}]} \exp\{-\frac{E_i}{T}\}$$

$$\begin{aligned} T_e &= T_i = T_n = T \\ E_i &= \text{ionisation energy} \end{aligned}$$

and defining $\alpha = \frac{n_e}{n_e+n_n}$ the relative degree of ionisation :

$$\alpha = \frac{1}{1 + \frac{n_i\, e^{E_i/T}}{3 \cdot 10^{27}\, T^{3/2}}}$$

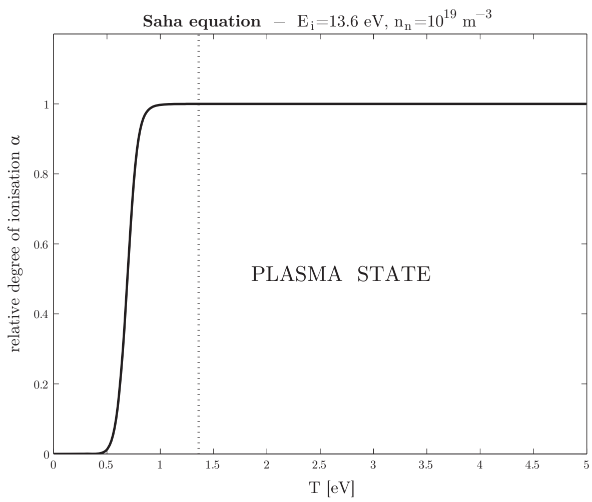


Fig. 5: α against temperature

Sharp transition from exponential → “phase transition” → plasma ≡ 4^{th} state of matter

Various plasmas

Ex. 1 : Air ($\sim N_2$, $E_i = 14.5$ eV), $T \sim 1/40$ eV, $n_n \approx 10^{25}$ m^{-3}, $\frac{n_e}{n_n} \sim \alpha \sim 10^{-120} \sim 0$!

Ex. 2 : Solar corona $T \sim 500$ eV, $n_e \sim 10^{13}$ m^{-3} ⇒ $\alpha \sim 1$

1.2 “Globally neutral” (‘quasi-neutrality’)

$n_e \approx n_i$ at least on average (both spatially and temporally). In the exercise you have demonstrated ‘quantitatively’ how difficult it is to violate quasi-neutrality.

Naturally, if we look at a microscopic level, this neutrality will be violated, for e.g., if we approach an individual ion. So, the question is, how close should we be to the single charge to feel its field ?

This question leads us to the 3^{rd} aspect of the definition of a plasma : collective effects.

1.3 "Collective effects"

1.3.1 Static : charge (or potential) screening

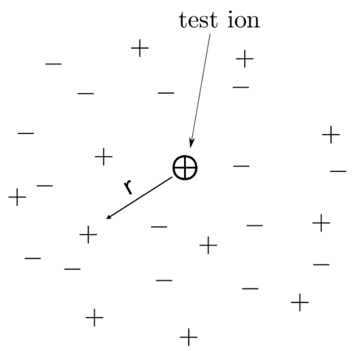


Fig. 6: An extra ion is inserted

Insert an extra positive charge (fig. 6) :

- potential around it ?
- density of electrons around it ?
- up to which distance perturbation will be felt ?

Solution in simple situation :

- $T_i = 0$ (ions don't move); $n_i = n_0$, and $n_e = n_0$ for $r \to \infty$
- Electrons are distributed according to Maxwell-Boltzmann : $n_e = n_0 \exp\{-\frac{\text{energy}}{T}\} = n_0 \exp\{\frac{e\phi(r)}{T}\}$
- Perturbation is small [quasi-neutrality !] : $\frac{e\phi}{T} \ll 1$
- singly ionised ions ($q_i = +e$)

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0} \Rightarrow \nabla^2 \phi = -\frac{\rho}{\varepsilon_0}$$

but

$$\rho = e(n_i - n_e) = en_0\{1 - e^{e\phi/T}\} \approx en_0\{1 - 1 - \frac{e\phi}{T} + \ldots\} = -\frac{e^2 n_0}{T}\,\phi$$

$$\Rightarrow \boxed{\nabla^2 \phi = \frac{e^2 n_0}{\varepsilon_0 T}\,\phi}$$

Note 1 : $\left(\sqrt{\frac{e^2 n_0}{\varepsilon_0 T}}\right)^{-1}$ has dimensions of a length. Define $\lambda_D = \sqrt{\frac{\varepsilon_0 T}{e^2 n_0}}$ "Debye length".

Note 2 : Spherical coordinates, only r dependence (problem is spherically symmetric).

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)$$

Substitute

$$U(r) = \phi(r) r$$

Calculation

$$\begin{aligned}
\frac{\partial\phi}{\partial r} &= \frac{\partial}{\partial r}\left(\frac{1}{r}U(r)\right) = -\frac{1}{r^2}U(r) + \frac{1}{r}\frac{\partial U}{\partial r} \\
\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left[-\frac{1}{r^2}U + \frac{1}{r}\frac{\partial U}{\partial r}\right]\right\} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left\{-U + r\frac{\partial U}{\partial r}\right\} = \frac{1}{r^2}\left\{-\frac{\partial U}{\partial r} + \frac{\partial U}{\partial r} + r\frac{\partial^2 U}{\partial r^2}\right\} \\
&= \frac{1}{r}\frac{\partial^2 U}{\partial r^2} = \frac{U}{r}\frac{1}{\lambda_D^2} \Rightarrow \boxed{\frac{1}{\not r}\frac{\partial^2 U}{\partial r^2} = \frac{U}{\not r}\frac{1}{\lambda_D^2}}
\end{aligned}$$

<u>Solution</u> :

$$\begin{aligned}
U(r) &= Ae^{-r/\lambda_D} + Be^{r/\lambda_D} \\
\text{or } \phi(r) &= A\frac{e^{-r/\lambda_D}}{r} + B\frac{e^{r/\lambda_D}}{r}
\end{aligned}$$

Boundary conditions :

$$\begin{cases}
\phi(r) \overset{r\to\infty}{\to} 0 & \Rightarrow \quad B = 0 \\
\underbrace{\phi(r) \overset{r\to 0}{\sim} \frac{e}{4\pi\varepsilon_0 r}}_{\text{single particle potential}} & \Rightarrow \quad A = \frac{e}{4\pi\varepsilon_0}
\end{cases}$$

$$\boxed{\Rightarrow \phi(r) = \frac{e}{4\pi\varepsilon_0}\frac{1}{r}\cdot e^{-r/\lambda_D}}$$

[e^{-r/λ_D} : term due to plasma collective interactions]

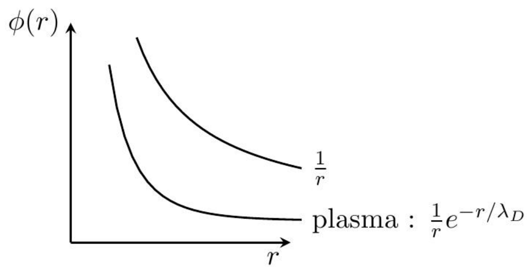


Fig. 7: $\phi(r)$

with

$$\lambda_D = \sqrt{\frac{\varepsilon_0 T}{e^2 n}} \begin{array}{ll} \leftarrow & \text{thermal motion makes shielding difficult} \\ \leftarrow & \text{space charge helps shielding} \end{array}$$

Density :

$$n_e(r) \cong n_0(1 + \frac{e\phi}{T}) = n_0\left\{1 + \underbrace{\frac{e^2}{4\pi\varepsilon_0 T}\frac{1}{r}e^{-r/\lambda_D}}_{>0 \text{ as expected}}\right\}$$

there are slightly more electrons around $\oplus$, but this extra electron density goes to 0 as $r \to \infty$, with λ_D as exponential decay length

<u>Note</u> : in order for this effect to happen, i.e. for the plasma to exhibit collective effects, we must have:

- enough particles within the sphere with radius equal to λ_D : $N_D = \frac{4}{3}\pi\lambda_D^3 n_0 \gg 1$
- plasma size $> \lambda_D$

1.3.2 Dynamical effect - Plasma Oscillations

By combining characteristic $\nearrow$ velocity $(v = \sqrt{T_e/m_e})$, $\searrow$ length (λ_d), we can find a characteristic time (or frequency) for the plasma response to local violation of quasi-neutrality :

$$\omega_p \sim \frac{1}{\tau} \sim \frac{v}{\lambda_D} = \sqrt{\frac{e^2 n_e}{\epsilon_0 m_e}}$$

Numerically :

$$f_p = \frac{\omega_p}{2\pi} \simeq 9000\sqrt{n\,[\mathrm{cm}^{-3}]}\ \mathrm{Hz}$$

<u>Obs.</u> : To propagate a wave in a simple plasma (no B-field) one should have $\omega > \omega_p$.

2 General problem of plasma description

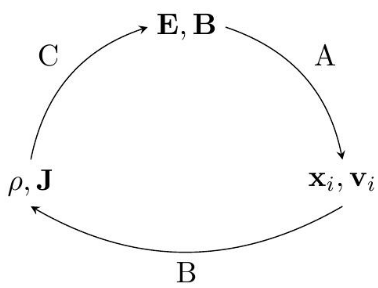


Fig. 8: "Typical self-consistent loop which appears in plasma physics."

How to combine the three steps ? We need a <u>model</u> to describe the plasma.

<u>Three models</u> :

1. Single particle (eq. of motion) { orbits; magnetic confinement }
2. Kinetic Boltzmann (Vlasov) { transport; wave-particle interactions }
3. Fluid (eq. motion, continuity, eq. of state) { macro-stability (equilibrium); waves }

Today we discuss the single-particle model, in particular step A of Fig.8, i.e. how $\mathbf{E}, \mathbf{B}$ affect particle motion.

Eq. of motion (non relativistic) :

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{F}_{\text{ext}} \quad \Longrightarrow \quad \begin{cases} \mathbf{x}(t) \\ \mathbf{v}(t) \end{cases} \quad \text{for given} \begin{cases} \mathbf{E}(\mathbf{x}, t) \\ \mathbf{B}(\mathbf{x}, t) \end{cases}$$

Different important cases:

1. $\mathbf{B} = \mathbf{B}_0$; $\mathbf{E} = 0$ Larmor motion (diamagnetism)
2. $\mathbf{B} = \mathbf{B}_0$; $\mathbf{E} = \mathbf{E}_0$ "$\mathbf{E} \times \mathbf{B}$ drift"
3. $\mathbf{B} = \mathbf{B}(\mathbf{x})$; $\mathbf{E} = 0$ "∇B" and curvature drift
   with $\nabla B \perp \mathbf{B}$ (variation across $\mathbf{B}$)
4. $\mathbf{B} = \mathbf{B}(\mathbf{x})$; $\mathbf{E} = 0$
   with $\nabla B \parallel \mathbf{B}$ (variation along $\mathbf{B}$) $\rightarrow$ magnetic mirror

Naturally, there exist other, more complicated situations, but the main aspects are captured in the four simple cases illustrated here.

2.1 $\mathbf{B}_0$ uniform, no $\mathbf{E}$

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\mathbf{v} \times \mathbf{B}_0$$

Conventional case; we won't repeat the equations.

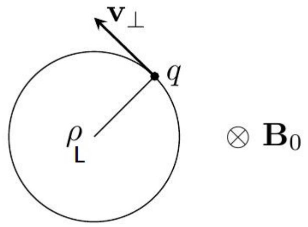


Fig. 9: Circular motion

$$\frac{mv_\perp^2}{\rho_L} = m\rho_L\Omega^2 = |q|\, v_\perp B_0 \quad \rightarrow \quad \begin{cases} \rho_L = \dfrac{v_\perp}{\Omega} \\ \Omega = \dfrac{|q|\, B_0}{m} \end{cases} \quad \text{(from laws of circular motion, } v_\perp = \rho\Omega\text{)}$$

Numerical values :

$$f = \frac{\Omega}{2\pi} = \begin{cases} \text{el.} & 28 \times B[\text{T}]\ [\text{GHz}] & \text{ex : ITER } f_e \sim 170 \text{ GHz} \\ \text{ions} & 15.2\frac{Z}{A} \times B[\text{T}]\ [\text{MHz}] & \text{ex : ITER } f_i \sim 50 \text{ MHz} \end{cases}$$

$$\rho_L = \frac{v_\perp}{\Omega} = \begin{cases} \text{el.} & 10^{-4}\frac{\sqrt{T_e\ [\text{keV}]}}{B_0\ [\text{T}]}\ [\text{m}] \qquad (\text{ex : 10 keV, B=3T} \rightarrow \rho_{L_e} \simeq 0.1\,\text{mm}) \\ \text{ions} & 5\cdot 10^{-3}\frac{\sqrt{T_e\ [\text{keV}]}}{B_0\ [\text{T}]}\sqrt{\frac{m_i}{m_p}}\ [\text{m}] \quad (\text{ex : 10 keV, B=3T, H-ions} \rightarrow \rho_{L_i} \simeq 7\,\text{mm}) \end{cases}$$

Obs. 1 : from our "design of reactor" study we know that $B \sim 5\,\text{T}$, plasma radius $>> 7\,\text{mm}$ so we are ok.

Obs. 2 :

- Energy is conserved (force is $\perp$ to $\mathbf{v}$: no work !) NB : **we neglect losses by radiation**
- Parallel and perpendicular motions are decoupled

Obs. 3 : sign of gyro-motion gives reduction of $\mathbf{B}_0$: **diagmagnetism** [remember : this is why $\beta = \frac{nT}{B^2/2\mu_0} < 1$]

2.2 $\mathbf{B}_0$ and $\mathbf{E}_0$ uniform

$\parallel$ motion : $m\frac{dv_\parallel}{dt} = qE_\parallel$
$$\begin{cases} v_\parallel(t) = v_{\parallel 0} + \frac{q}{m}E_0 t & \text{(nothing special)} \\ z(t) = z_0 + v_{\parallel 0}t + \frac{1}{2}\frac{q}{m}E_0 t^2 \end{cases}$$

$\perp$ motion : sketch for $\mathbf{E}_0 \perp \mathbf{B}_0$:

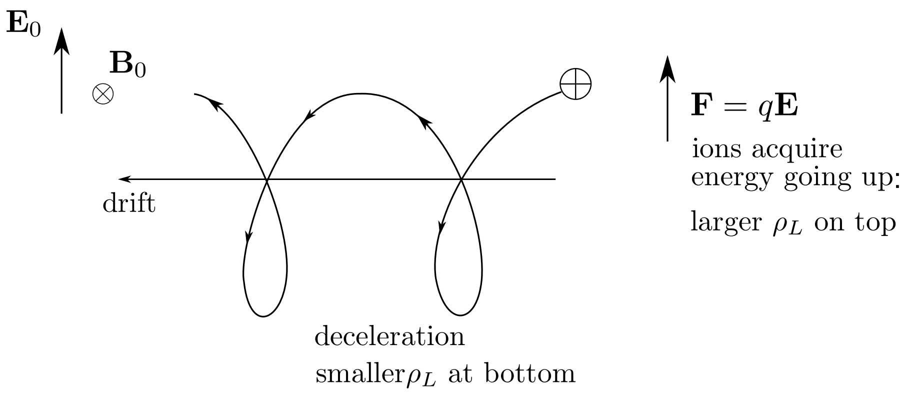


Fig. 10: Motion of the ions

And the same holds for electrons

$\Rightarrow$ **Drifts to the left for both species.**

How to calculate this steady-state drift ?

Def : $\mathbf{v}_L$ = solution of case $\mathbf{B} = \mathbf{B}_0$; $\mathbf{E} = 0$ (Larmor motion)

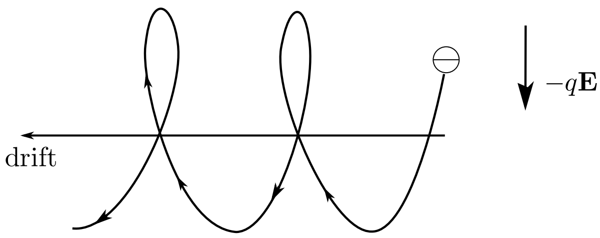


Fig. 11: Motion of the electrons

We look for $\mathbf{v}_d$, such that $\mathbf{E}_0 + \mathbf{v}_d \times \mathbf{B}_0 = 0$; in fact $\mathbf{v}_L + \mathbf{v}_d$ is the solution ($\mathbf{v}_d$ = constant).

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q(\mathbf{E}_0 + \mathbf{v} \times \mathbf{B}_0);\ \text{but } \mathbf{v}_L : m\frac{\mathrm{d}\mathbf{v}_L}{\mathrm{d}t} = q\mathbf{v}_L \times \mathbf{B}_0$$

$$m\frac{\mathrm{d}\mathbf{v}_L}{\mathrm{d}t} + m\frac{\mathrm{d}\mathbf{v}_d}{\mathrm{d}t} = q\mathbf{E}_0 + q\mathbf{v}_L \times \mathbf{B}_0 + q\mathbf{v}_d \times \mathbf{B}_0$$

So, we need to solve[1] $\boxed{\mathbf{E}_0 + \mathbf{v}_d \times \mathbf{B}_0 = 0}$ $\qquad (\times \mathbf{B}_0)$

$$\mathbf{E}_0 \times \mathbf{B}_0 + (\mathbf{v}_d \times \mathbf{B}_0) \times \mathbf{B}_0 = 0 \quad \Rightarrow \quad \mathbf{E}_0 \times \mathbf{B}_0 + \underbrace{(\mathbf{v}_d \cdot \mathbf{B}_0)\mathbf{B}_0}_{\parallel \text{ to } B_0} - \mathbf{v}_d B_0^2 = 0$$

$\perp$ motion (the one we are interested in) :

$$\mathbf{E}_0 \times \mathbf{B}_0 = \mathbf{v}_{d_\perp} B_0^2 \quad \Rightarrow \quad \boxed{\mathbf{v}_{d_\perp} = \mathbf{v}_{E\times B} = \frac{\mathbf{E}_0 \times \mathbf{B}_0}{B_0^2}}$$

→ “E cross B drift” (of “guiding center”) as expected from sketch.

Obs. :

- $\mathbf{v}_d$ is independent of charge, m, v.
- same for ions, electrons → no current, no charge separation

Generalisation of $\mathbf{v}_{E\times B}$

Instead of $q\mathbf{E}_0$, we can put any force $\mathbf{F}_{\text{ext}} = q\mathbf{E}_0$, or $\mathbf{E}_0 = \frac{\mathrm{F}_{\text{ext}}}{q}$

$$\Rightarrow \mathbf{v}_F = \frac{\mathbf{F}_{\text{ext}}/q \times \mathbf{B}_0}{B_0^2} \qquad \text{ex : gravity : } \mathbf{v}_g = \frac{m\mathbf{g}_{\text{ext}}/q \times \mathbf{B}_0}{B_0^2} \tag{2.1}$$

Obs. : if $\mathbf{F}_{\text{ext}}$ does **not** contain the charge, then drift **does** depend on charge (→ charge separation)

[1] $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \times \mathbf{b}) \times \mathbf{a} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

2.3 $\mathbf{B}(\mathbf{x})$ and $\mathbf{E} = 0$, with $\nabla B \perp \mathbf{B}$ (variation across B)

Two possibilities :

(3a) $|\mathbf{B}|$ changes, but **not** the direction

(3b) only the direction of $\mathbf{B}$ changes ($\mathbf{B}/|\mathbf{B}|$ changes along $\mathbf{B}$)

(3a)

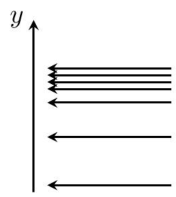


Fig. 12: $|\mathbf{B}|$ increases with y (vertically)

Calculation is complicated, but if the variations are not too fast (in space and in time), we can always decompose the motion into $\begin{cases} \text{Larmor motion} \\ \text{drift of guiding center} \end{cases}$

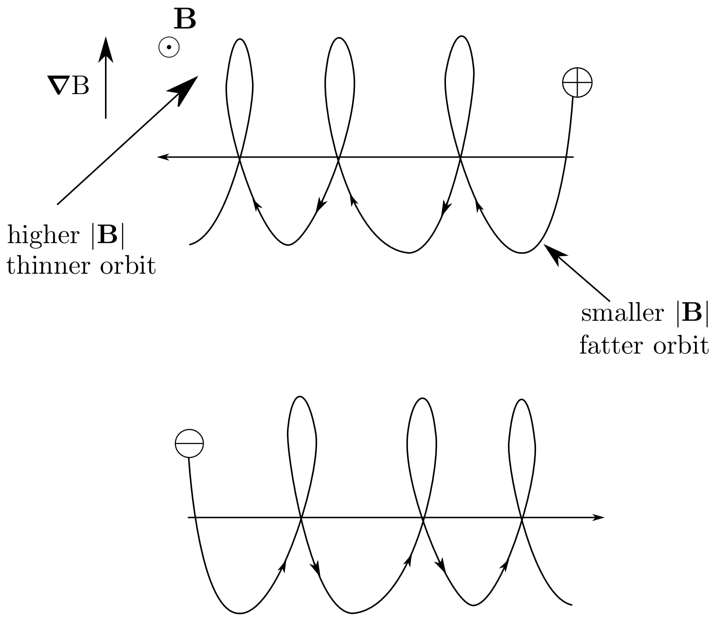
Conditions :

$$L_\perp = \left|\frac{B}{\nabla_\perp B}\right| \gg \rho_L \quad \text{variation occurs over distances larger than } \rho_L$$

$$L_\parallel = \left|\frac{B}{\nabla_\parallel B}\right| \gg v_\parallel \underbrace{\frac{2\pi}{\Omega}}_{\text{space covered in one Larmor orbit}}$$

$$\text{time} \qquad \left|\frac{1}{B}\frac{\partial B}{\partial t}\right| \ll \Omega$$

Sketch :



Drift should be to the
$\begin{cases} \text{left} & \text{for ions} \\ \text{right} & \text{for electrons} \end{cases}$

$\mathbf{v} \sim q\mathbf{B} \times \nabla B$

Fig. 13: Motion of ions (upper) and electrons (lower).

Precise calculation gives in fact

$$\boxed{\mathbf{v}_d = \frac{1}{2q} m \mathbf{v}_\perp^2 \frac{\mathbf{B} \times \nabla B}{B^3}} = \mathbf{v}_{\nabla B} \qquad \text{“grad B drift”}$$

<u>Obs.</u> : $\mathbf{v}_{\nabla B}$ depends on q

$\Rightarrow$ current

$\Rightarrow$ charge separation

(3b) Variation in direction of $\mathbf{B}$

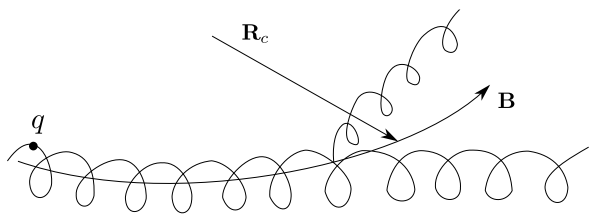


Fig. 14: Sketch

Guiding center feels centrifugal force :

$$\mathbf{F}_{\text{gc}} = m \frac{v_\parallel^2}{R_c} \frac{\mathbf{R}_c}{R_c} \equiv \mathbf{F}_{\text{ext}}$$

$\Rightarrow$ we know $\mathbf{v}_d = \dfrac{1}{q}\dfrac{\mathbf{F}_{\text{ext}} \times \mathbf{B}}{B^2}$. So the "curvature drift" is

$$\mathbf{v}_{\text{curv}} = \frac{1}{q\,B^2}\, m\frac{v_\parallel^2}{R_c}\,\frac{\mathbf{R}_c}{R_c} \times \mathbf{B} = \frac{1}{q}\,\frac{mv_\parallel^2}{R_c^2}\,\frac{\mathbf{R}_c \times \mathbf{B}}{B^2}$$

Case of vacuum fields (how to confine a plasma ?)

- Combine $\mathbf{v}_{\nabla B}$ and $\mathbf{v}_{\text{curv}}$
- In vacuum (in cyl. coordinates) $|\mathbf{B}| \propto \dfrac{1}{R_c}$ so that

$$\frac{\nabla B}{B} = -\frac{1}{R_c^2}\,\mathbf{R}_c$$

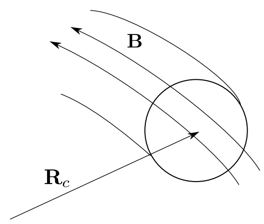


Fig. 15: Confining a plasma

So

$$\begin{aligned}
\mathbf{v}_d^{\text{total}} &= \frac{1}{2q} m v_\perp^2\,\frac{\mathbf{B} \times \nabla B}{B^3} + \frac{1}{q} m v_\parallel^2\,\frac{\mathbf{R}_c \times \mathbf{B}}{B^2 R_c^2} \\
&= \frac{1}{2q} m v_\perp^2\,\frac{\mathbf{R}_c \times \mathbf{B}}{B^2 R_c^2} + \frac{1}{q} m v_\parallel^2\,\frac{\mathbf{R}_c \times \mathbf{B}}{B^2 R_c^2} \\
&= \left[\frac{1}{2} m v_\perp^2 + m v_\parallel^2\right]\,\frac{1}{q}\,\frac{\mathbf{R}_c \times \mathbf{B}}{B^2 R_c^2} \\
&= \text{if we average over distribution :} \\
&\quad < \frac{1}{2} m v_\perp^2 >= T \qquad \text{(2 degrees of freedom)} \\
&\quad < \frac{1}{2} m v_\parallel^2 >= \frac{T}{2} \qquad \text{(1 degree of freedom)}
\end{aligned}$$

$$\Rightarrow \boxed{< \mathbf{v}_d^{\text{total}} >= \frac{2T}{q}\,\frac{\mathbf{R}_c \times \mathbf{B}}{B^2 R_c^2}}$$

Obs. :

- $\mathbf{v}_{\text{curv}}$ and $\mathbf{v}_{\nabla B}$ add up (unfortunately !)
- dependence on q $\rightarrow$ charge separation
- Proportional to energy, no mass dependence

Magnitude :

$$\left|< \mathbf{v}_d^{\text{total}} >\right| = \frac{2T}{|q|}\,\frac{1}{B R_c} = v\,\frac{\rho_L}{R_c}$$

Can we conclude anything about particle confinement ?

Simplest idea : particles move freely along **B** but **not** acroos : let's confine them in a **toroidal** system (no beginning, no end).

What happens ?
[This is done in exercices]

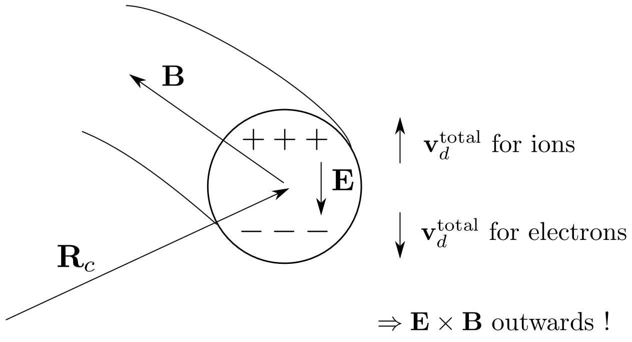


Fig. 16: **E** × **B** outwards ⇒ **no confinement**

We need a more complicated **B**-field structure to confine particles ! (→ rotational transform, tokamak, . . .)

2.4 B(x) and E = 0, with $\nabla B \parallel$ B (variation of |B| along B)

$\nabla B \parallel \mathbf{B}$

$$L_\parallel = \left|\frac{B}{\nabla_\parallel B}\right| \gg \underbrace{v_\parallel \frac{2\pi}{\Omega}}_{\text{space covered in one Larmor orbit}} \quad \textbf{(to have Larmor motion)}$$

Cylindrical coordinates (r, θ, z) :

- $B_\theta = 0 \rightarrow B_z = B_z(z)$
- but $B_r \neq 0$ (field lines are not exactly along z)

$$\nabla\cdot\mathbf{B} = 0 \overset{\text{cyl. coord.}}{\Rightarrow} \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0 \quad \rightarrow \frac{\partial}{\partial r}(rB_r) = -r\frac{\partial B_z}{\partial z}$$

$$rB_r = -\int_0^r \mathrm{d}r' \left(r'\frac{\partial B_z}{\partial z}\right) \overset{\text{small distances}}{\simeq} \left(-\left.\frac{\partial B_z}{\partial z}\right|_{r=0}\right)\frac{r^2}{2} \quad \Rightarrow B_r \simeq -\frac{r}{2}\left.\frac{\partial B_z}{\partial z}\right|_{r=0}$$

Obs. : $|B| = |B(r,z)| \quad \rightarrow \quad \mathbf{B}\times\nabla B$ drift is azimuthal (not important).

We are interested in **parallel** motion.

Consider a particle whose guiding center lies on $z = 0$ axis : $r \sim \rho_L$

$$\boxed{B_r \simeq -\frac{\rho_L}{2}\frac{\partial B_z}{\partial z}} \{\text{“little” } \mathrm{B_r} \text{ produced by the fact that B varies with z}\}$$

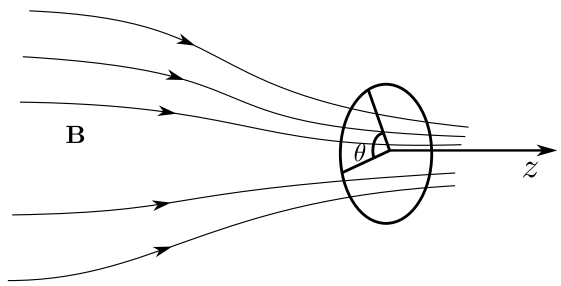


Fig. 17: Description with cylindrical coordinates

which parallel force does this produce ?

Parallel force (with $B_\theta = 0$ and $v_\theta = \mp v_\perp = -\frac{|q|}{q} v_\perp$):

$$\begin{aligned} q(\mathbf{v}\times\mathbf{B})_z &= q(v_r B_\theta - v_\theta B_r) = -q(\mp v_\perp)B_r = +q\frac{|q|}{q} v_\perp B_r \\ &= |q|\, v_\perp B_r \simeq |q|\, v_\perp \left(-\frac{\rho_L}{2}\frac{\partial B_z}{\partial z}\right) = -\,|q|\, v_\perp \underbrace{\frac{v_\perp m}{|q|\, B}}_{\rho_L} \frac{1}{2}\frac{\partial B_z}{\partial z} \\ &= -\underbrace{\frac{1}{2}\frac{v_\perp^2 m}{B}}_{\mu \text{ “\textbf{magnetic moment}” (corresponds to current}\times\text{area)}} \frac{\partial B_z}{\partial z} = -\mu\frac{\partial B_z}{\partial z} \end{aligned}$$

So, both electrons and ions feel a force that tries to prevent them from moving towards **increasing** field.

Eq. of motion ($\parallel$ to $\mathbf{B}$) :

$$\boxed{m\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = -\mu\nabla_\parallel B}$$

This is a very useful form, as, in the conditions of interest here, μ **is conserved** [“**adiabatic invariant**”, i.e. constant if the changes in the system are slow (in time and space)].

μ is constant along particle motion : as $\mu = \frac{1/2 m v_\perp^2}{B}$, if B increases, $v_\perp$ must increase.

But total kinetic energy is constant, therefore $v_\perp^2 + v_\parallel^2 = \text{const} \quad \Rightarrow \quad$ if $v_\perp$ increases, $v_\parallel$ must decrease.

Magnetic mirror

$$\frac{m v_\parallel^2}{2} = \text{ Energy} - \mu B$$

If $\mathbf{B}$ is large enough in the throat, $v_\parallel \to 0$ and can change sign [both for ions and electrons] $\Rightarrow$ particles are “reflected”

Obs. 1 : This is how particles are confined in Van-Allen belts in Earth dipole field.

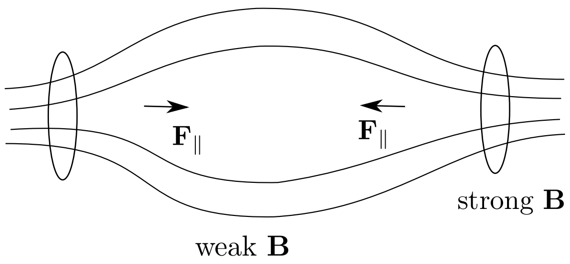


Fig. 18: Magnetic mirror

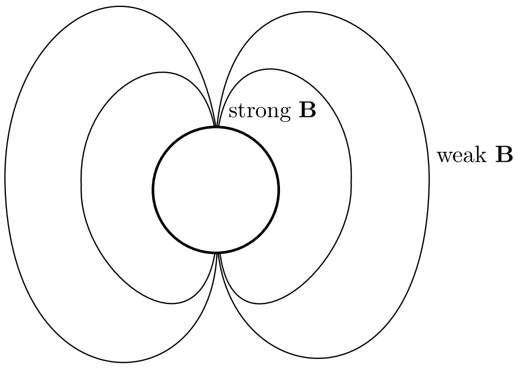


Fig. 19: Earth dipole field

[ions from thermonuclear explosions stay confined for many years !]

<u>Obs. 2</u> : Magnetic mirrors were the first confinement schemes to be tested for fusion. But they have a **problem** : not all particles are confined ! E.g. : if $v_\perp = 0 \to \mu = 0 \to F_{\parallel} = 0$. One needs a large enough ratio $v_\perp / v_{\parallel}$.

In the exercice today you'll evaluate this limit ("loss cone").