Nuclear Fusion and Plasma Physics **EPFL**
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Solutions to Problem Set 2

Exercise 1 - Perfect Plasma Power Reactor

a) We must try to minimize the quantity

$$F = \frac{\rho_{blanket} V_{blanket}}{P_E} = \frac{\rho_{bl}}{P_E} 2\pi^2 R_0 \left[(b+a)^2 - a^2\right] = \frac{\rho_{bl}\, 2\pi^2}{P_E} R_0 (b^2 + 2ab)$$

which is a function of R_0 and a, under the constraints

$$P_E = \frac{1}{4}\eta_t(\Delta E_{fus} + \Delta E_{Li})\, n^2 \langle\sigma v\rangle_{D-T} (2\pi^2 R_0 a^2) = 1\,\mathrm{GW} \tag{1}$$

$$L_W = \frac{\frac{1}{4}(\Delta E_\alpha + 0.3\,\Delta E_n)\, n^2 \langle\sigma v\rangle_{D-T} (2\pi^2 R_0 a^2)}{(4\pi^2 R_0 a)} < 4\,\mathrm{MW/m^2} \tag{2}$$

The second constraint can be written as

$$n^2 \langle\sigma v\rangle_{D-T}\, a = \frac{8 L_W^{max}}{(\Delta E_\alpha + 0.3\,\Delta E_n)}, \text{ where } L_W^{max} = 4\ \mathrm{MW/m^2} \tag{3}$$

Replacing in the equation for the first constraint gives:

$$P_E = 2\,\eta_t \frac{\Delta E_{fus} + \Delta E_{Li}}{\Delta E_\alpha + 0.3\,\Delta E_n} L_W^{max} (2\pi^2 R_0 a) \to R_0 = \frac{1}{4\pi^2 \eta_t} \frac{P_E}{L_W^{max}} \frac{\Delta E_\alpha + 0.3\,\Delta E_n}{\Delta E_{fus} + \Delta E_{Li}} \frac{1}{a} \tag{4}$$

Now we use this result in the expression for F, to express the quantity to be minimized as

$$F = \frac{\rho_{bl}}{2\eta_t} \frac{1}{L_W^{max}} \frac{\Delta E_\alpha + 0.3\,\Delta E_n}{\Delta E_{fus} + \Delta E_{Li}} \left(\frac{b^2}{a} + 2b\right) \tag{5}$$

F is minimized for $a \to \infty$ and correspondingly $R_0 \to 0$. This is limited by the topological constraint $a < R_0 - b$: the plasma minor radius cannot be larger than the torus major radius minus the thickness of the blanket. So the minimum value of F is reached for $a = R_0 - b$. Replacing $R_0 = a + b$ in (4), we get a quadratic equation for a :

$$a^2 + ba - \frac{1}{4\pi^2 \eta_t} \frac{P_E}{L_W^{max}} \frac{\Delta E_\alpha + 0.3\,\Delta E_n}{\Delta E_{fus} + \Delta E_{Li}} = 0.$$

We are interested in the positive solution :

$$\begin{aligned} a &= \frac{-b + \sqrt{b^2 - 4\left(-\frac{1}{4\pi^2\eta_t}\frac{P_E}{L_W^{max}}\frac{\Delta E_\alpha + 0.3\,\Delta E_n}{\Delta E_{fus} + \Delta E_{Li}}\right)}}{2} \\ &= \frac{-1.5\mathrm{m} + \sqrt{(1.5\mathrm{m})^2 + \frac{1}{\pi^2 \cdot 0.35}\frac{1\cdot 10^9\mathrm{W}}{4\cdot 10^6 \mathrm{W/m^2}}\frac{(3.5+4.2)\mathrm{MeV}}{22.4\mathrm{MeV}}}}{2} = 1.85\,\mathrm{m}. \end{aligned}$$

The major radius is then

$$R_0 = a + b = 3.35\,\mathrm{m}$$

and the corresponding value of F is

$$F_{min} = \frac{3 \cdot 10^3\,\mathrm{kg/m^3}}{2 \cdot 0.35} \frac{1}{4\mathrm{MW/m^2}} \frac{(3.5 + 4.2)\mathrm{MeV}}{22.4\,\mathrm{MeV}} \left(\frac{(1.5\,\mathrm{m})^2}{1.85\,\mathrm{m}} + 2 \cdot 1.5\,\mathrm{m} \right) \approx 1.56{\times}10^3\,\mathrm{kg/MW}.$$

This makes the mass utilization factor be ≈ 1.56 times that of a fission reactor.

Exercise 2 - Plasma β limit and diamagnetism

a) The current per unit length generated by a single charged particle in its Larmor orbit (often referred to as the Larmor "circuit") is given by $i_j = -q_j \frac{\Omega_j}{2\pi}$, where Ω_j is the cyclotron frequency of the particle. This current produces a magnetic field according to Ampère's law, which in differential form is $\nabla \times \vec{B} = \mu_0 \vec{J}$, where $\vec{J}$ is the current density.

   Using this relationship, we can express the small change in the magnetic field δB as $\delta B = \mu_0 \delta i$, where δi is the change in current produced by the particles. Substituting the expression for the current i_j, we obtain $\delta B = -\frac{\mu_0 q_j^2 B}{2\pi m_j} \delta n$, where δn represents the change in particle density.

   This result shows that the induced magnetic field δB is negative, meaning it opposes the original magnetic field B. This induced field has the same sign for both electrons and ions, which implies that the motion of charged particles in a magnetic field tends to create a magnetic field that counteracts and reduces the strength of the externally applied magnetic field. This phenomenon is the basis of plasma diamagnetism.

b) At a given point in space, the number of particles that contribute to the reduction of the B field is determined by the number of particles whose Larmor orbit passes through that point. This contribution is proportional to the particle density n and the surface area swept by each Larmor orbit, given by $S_L = \pi \rho_L^2 \sim v_\perp^2$. Since $v_\perp^2$ is proportional to the plasma's thermal kinetic energy, and thus to the temperature T, the overall effect on the magnetic field is proportional to both n and T. Therefore, the induced diamagnetic field is proportional to the plasma pressure, leading to the relationship $\delta B \sim nT$.

   Here we have assumed that $v_\perp$ is equal for all particles at all points in space. In reality we would have to integrate over the distribution function $f(x, v)$ describing the distribution of velocity over the particle population: $\delta B(x) = -\frac{\mu_0}{B} \int \frac{1}{2} m_j v_\perp^2 f(x, v) d^3v = -\frac{\mu_0 n T}{B}$.

c) At some point, the diamagnetism will reduce the magnetic field to such an extent that the Larmor radius becomes larger than the machine size and the particles are no longer confined.

Exercise 3 - Violating quasi-neutrality

In the definition of a plasma, it is stated that, although it is an ionized gas made of separate positive and negative charges, it is globally neutral.

However, if this balance is disturbed, even slightly, significant forces can arise within the plasma. In this exercise, we will explore the consequences of a small violation of quasi-neutrality, where the density of positive charges slightly exceeds that of negative charges. Such a scenario can provide insights into the forces that work to restore equilibrium in a plasma.

a) We first set up our simple 1D model of a plasma by considering a plasma located between $x = -d/2$ and $x = d/2$, where $d = 1\,\text{m}$. It is assumed homogeneous and infinite in all other directions (y and z), which simplifies our calculations by focusing on the essential physics in one dimension.

The 1% violation of quasineutrality means that we will have a surplus of $0.01\,n$ ions. This excess of positive charge creates an imbalance that leads to the development of an electric field within the plasma. This will result in a charge density (charge per unit volume) $\rho = 0.01\,e\,n$. First, we write Poisson's equation in one dimension, which relates the spatial derivative of the electric field to the charge density:

$$\frac{\partial E}{\partial x} = \frac{\rho}{\epsilon_0} \tag{6}$$

The electric field in the plasma can be found by integrating this differential equation over the plasma, which is straightforward since we assumed the violation in quasineutrality—and hence the charge density—to be uniform.

$$\begin{aligned} E(x) &= \int \frac{\rho}{\epsilon_0}\,dx = \frac{\rho}{\epsilon_0}\int dx \\ &= \frac{\rho}{\epsilon_0}x + C \end{aligned}$$

Here C is an arbitrary constant of integration. To determine its value, we consider that at the exact center of the plasma, the charged particles experience no net electric field because the contributions from the surrounding particles on all sides cancel out. This symmetry implies that the electric field $E(x)$ must be zero at the center, i.e., $E(0) = 0$. Therefore, we set $C = 0$. The electrostatic force per unit volume, which is the product of the charge density ρ and the electric field $E(x)$, is then given by

$$F_e(x) = \rho E(x) = \frac{\rho^2}{\epsilon_0}x. \tag{7}$$

To find the magnitude of this force for our specific plasma parameters, we calculate

$$|F| = \frac{(0.01\,n\,e)^2 d}{\epsilon_0} = \frac{(0.01\cdot 10^{20}\,\text{m}^{-3})^2(1.6\times10^{-19}\text{C})^2\cdot 1.0\,\text{m}}{8.85\times10^{-12}\,\frac{\text{C}^2}{N\text{m}^2}} \sim 10^9\,\text{N/m}^3. \tag{8}$$

b) We now compare this electrostatic force to other forces acting on the plasma, such as gravitational and pressure forces:

- Gravity exerts a force per unit volume $F_g = \rho_m g$ where ρ_m is the mass density, so

$$\begin{aligned} F_g &= (m_e n_e + m_i n_i)\, ng \approx m_i ng \\ &\approx 10^{-27}\text{kg} \cdot 10^{20}\text{m}^{-3} \cdot 9.8\,\text{m/s}^2 \\ &\approx 10^{-6}\,\text{N/m}^3 \end{aligned}$$

- Pressure is given by $p = nT$ where we have to take care to convert the T into Joules. For our plasma this is $p = nT = 10^{20}\,\text{m}^{-3} \cdot 1.6 \times 10^{-19}\,\text{J/eV} \cdot 10 \times 10^3\,\text{eV} = 10^5\text{N/m}^2$. The force exerted per cubic meter of plasma is $p/1\text{m} = 10^5\,\text{N/m}^3$

These forces are several orders of magnitude smaller than the electric force trying to maintain quasineutrality, highlighting the dominant role of electrostatic forces in restoring plasma neutrality.
