Nuclear Fusion and Plasma Physics - Exercises

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Solutions to problem set 1 - 09 September 2024

Exercise 1 - Combustion, Fission or Fusion?

a) Each chemical reaction releases $Q_N = 6.5 \cdot 10^{-19}$J of energy. This energy is transported by the reaction products. A mixture of carbon and oxygen with a total weight of 1 kg releases

$$Q_m = Q_N \cdot N_A \cdot n = Q_N N_A \frac{m}{M_C + M_{O_2}} \tag{1}$$

where $N_A = 6.02 \cdot\ 10^{23}$ mol^{-1} is Avogadro's number, M_C and M_{O_2} the molar masses of carbon and oxygen. Note that for a molecule X, M_X/N_A is the mass per mole of X divided by the number of atoms/molecules of X in a mole, so this ratio corresponds to the mass m_X of a single atom/molecule of X. As a result, the formula above can also be understood as the energy per reaction multiplied by the ratio of fuel mass over the mass of the reactants in one reaction; this ratio is thus the number of reactions possible with the available amount of fuel.
With a power consumption $P = Q/t$ we will therefore have enough energy for

$$t = \frac{Q_m}{P} = \frac{Q_N N_A m}{P\,(M_C + M_{O_2})} = \tag{2}$$

$$= \frac{6.5 \cdot 10^{-19}\text{J} \cdot 6.02 \cdot 10^{23} 1/\text{mol} \cdot 10^3\text{g}}{\frac{18780 \cdot 10^3 \text{Wh/year} \cdot 3600\text{s/h}}{3.14 \cdot 10^7 \text{s/year}} \cdot 44\,\text{g/mol}} = 1.15\,\text{h}$$

b) In a traditional nuclear power plant, the absorption of a neutron (n) by a nucleus of uranium-235 causes the **fission** of the uranium nucleus into two lighter elements. The energy released by a single isotope of uranium-235 is

$$Q_N = \Delta m c^2 = \Big\{\ (m_n + m_U) - (m_{Ce} + m_{Zr} + 2m_n)\ \Big\}\, c^2 = \tag{3}$$

$$\Big\{\ (1.0087 \text{ u} + 235.04 \text{ u}) - (139.91 \text{ u} + 93.91 \text{ u} + 2 \cdot 1.0087 \text{ u})\ \Big\} \times$$

$$\times 1.66 \cdot 10^{-27}\text{kg} \cdot (3 \cdot 10^8 \text{m/s})^2 = 3.2 \cdot 10^{-11} \text{ J}$$

This energy is transported by the reaction products (in the form of kinetic energy). A mixture of U-235 and neutrons with a total mass of 1kg will release an energy of

$$Q_m = Q_N \cdot N_A \cdot n = Q_N \frac{N_A m}{M_U + M_n} \tag{4}$$

with $N_A = 6.02 \cdot\ 10^{23}$ mol^{-1} (Avogadro's number), M_U and M_n the molar masses of U-235 and of the neutron. For a power consumption of $P = Q/t$ there will therefore be enough energy for

$$t = \frac{Q_m}{P} = \frac{Q_N N_A m}{P\,(M_U + M_n)} = \tag{5}$$

$$= \frac{3.2 \cdot 10^{-11}\ \text{J}\ \cdot 6.02 \cdot 10^{23}\ 1/\text{mol} \cdot 10^3\ \text{g}}{\frac{18780 \cdot 10^3\ \text{Wh/year} \cdot 3600\text{s/h}}{3.14 \cdot 10^7 \text{s/year}} \cdot 236\ \text{g/mol}} = 1.2 \cdot 10^3\ \text{years}$$

c) The **fusion** of two hydrogen atoms, in our case the isotopes deuterium (D) and tritium (T), yields

$$Q_N = \Delta m c^2 = \Big\{\ (m_D + m_T) - (m_{He} + m_n)\ \Big\}\, c^2 = \tag{6}$$

$$\Big\{\ (2.014\ \text{u} + 3.0164\ \text{u}) - (4.0027\ \text{u} +\ \cdot 1.0087\ \text{u})\ \Big\} \times$$

$$\times 1.66 \cdot 10^{-27}\text{kg} \cdot (3 \cdot 10^8 \text{m/s})^2 = 2.8 \cdot 10^{-12}\ \text{J}$$

The energy released by 1kg of fusion fuel (a 50:50 mixture of D-T):

$$Q_m = Q_N \frac{N_A m}{M_D + M_T} \tag{7}$$

Which is sufficient for consumption over a period of

$$t = \frac{Q_m}{P} = \frac{Q_N N_A m}{P(M_D + M_T)} = \tag{8}$$

$$= \frac{2.8 \cdot 10^{-12}\ \text{J}\ \cdot 6.02 \cdot 10^{23}\ 1/\text{mol} \cdot 10^3\ \text{g}}{\frac{18780 \cdot 10^3\ \text{Wh/years} \cdot 3600\text{s/h}}{3.14 \cdot 10^7 \text{s/years}} \cdot 5\ \text{g/mol}} = 5.0 \cdot 10^3\ \text{years}$$

Exercise 2 - A "small" Tokamak

a) The fusion power for a reaction between fusion fuels can be calculated simply by multiplying the energy per reaction $\langle \Delta E \rangle$ times the reaction rate in the total volume: $(n_1 n_2 \langle \sigma v \rangle V)\Delta E$.

We can then calculate the fusion power for D-T reactions: $P_{DT} = P_\alpha + P_n$

$$\begin{aligned} P_\alpha &= n_D n_T \langle \sigma v \rangle_{DT} V \Delta E_\alpha \\ &= \left(\frac{1}{2} \times 10^{20}\mathrm{m}^{-3}\right)^2 \times 10^{-22}\frac{\mathrm{m}^3}{\mathrm{s}} \times 3.5 \times 10^6\mathrm{eV} \times 1.6 \times 10^{-19}\frac{\mathrm{J}}{\mathrm{eV}} \times 10\mathrm{m}^3 \simeq 1.4\,\mathrm{MW} \end{aligned} \tag{9}$$

$$\begin{aligned} P_n &= n_D n_T \langle \sigma v \rangle_{DT} V \Delta E_n \\ &= \left(\frac{1}{2} \times 10^{20}\mathrm{m}^{-3}\right)^2 \times 10^{-22}\frac{\mathrm{m}^3}{\mathrm{s}} \times 14 \times 10^6\mathrm{eV} \times 1.6 \times 10^{-19}\frac{\mathrm{J}}{\mathrm{eV}} \times 10\mathrm{m}^3 \simeq 5.6\,\mathrm{MW} \end{aligned} \tag{10}$$

$$P_{DT} = P_\alpha + P_n = 1.4\,\mathrm{MW} + 5.6\,\mathrm{MW} = 7\,\mathrm{MW}$$

For D-D reactions we need the average energy per reaction, which is simply $\frac{1}{2}(4 + 3.25)\mathrm{MeV} \simeq 3.625\mathrm{MeV}$.

$$\begin{aligned} P_{DD} &= \frac{1}{2} n_D n_D \langle \sigma v \rangle_{DD} V \langle \Delta E \rangle_{DD} \\ &= \frac{1}{2}\left(10^{20}\mathrm{m}^{-3}\right)^2 \times 10^{-24}\frac{\mathrm{m}^3}{\mathrm{s}} \times 3.625\mathrm{MeV} \times 1.6 \times 10^{-19}\frac{\mathrm{J}}{\mathrm{eV}} \times 10\mathrm{m}^3 \\ &\simeq 29\mathrm{kW} \end{aligned} \tag{11}$$

Here a factor of $\frac{1}{2}$ is introduced to avoid counting the same reaction twice.

b) We can now compute the physics fusion gains:

$$Q_{DT} = \frac{P_f}{P_{in}} = \frac{7\,\mathrm{MW}}{28\,\mathrm{MW}} = 25\% \tag{12}$$

$$Q_{DD} = \frac{P_f}{P_{in}} = \frac{14.5\,\mathrm{kW}}{28\,\mathrm{MW}} = 0.05\% \tag{13}$$

We already see that we are far from break-even (Q=1) for both cases.

The Lawson parameter for break-even at $T = 10\,\mathrm{keV}$ is $n_e \tau_E > 10^{20}\,\mathrm{s\,m}^{-3}$. The confinement time is given by

$$\tau_E = \frac{W_{tot}}{P_l} \tag{14}$$

where P_l is the losses power. To obtain P_l we consider that in steady-state (i.e. to keep a constant plasma energy) we have $P_l = P_{in} + P_\alpha$. However, we have seen in question a) that $P_\alpha = 1/5 P_f << P_{in}$, namely no significant fusion power is generated, so we can take $P_l \approx P_{in}$. The total energy W_{tot} can be calculated as:

$$W_{tot} = \frac{3}{2} n_e (T_e + T_i) V = \frac{3}{2} 10^{20} \mathrm{m}^{-3} \times 20 \times 10^3 \mathrm{eV} \times 1.6 \times 10^{-19} \frac{\mathrm{J}}{\mathrm{eV}} \times 10\,\mathrm{m}^3 = 4.8\,\mathrm{MJ}$$

We can calculate the confinement time:

$$\tau_E = \frac{W_{tot}}{P_l} \simeq \frac{W_{tot}}{P_{in}} = \frac{4.8\,\mathrm{MJ}}{28\,\mathrm{MW}} = 0.17\,\mathrm{s}$$

so

$$n_e \tau_E = 1.7 \times 10^{19}\,\mathrm{m}^{-3}\,\mathrm{s}$$

Which is about one order of magnitude lower than the break-even requirement.

Note: In the presence of a significant α-particle fraction we should have considered that they contribute to the population of positively charged particles. In that case, the quasi-neutrality condition would be $n_\alpha + n_T + n_D = n_e$ instead of $n_T + n_D = n_e$.

c) In order to calculate the wall loading ($\mathrm{MW/m^2}$) of our "small" tokamak, one needs first to find the surface area in which the total power will be distributed. The surface area of a torus with a circular cross-section is given by

$$A_{torus} = 4\pi^2 r R_0$$

where r is the minor radius of the torus, while the volume of a torus is given by

$$V_{torus} = 2\pi^2 r^2 R_0$$

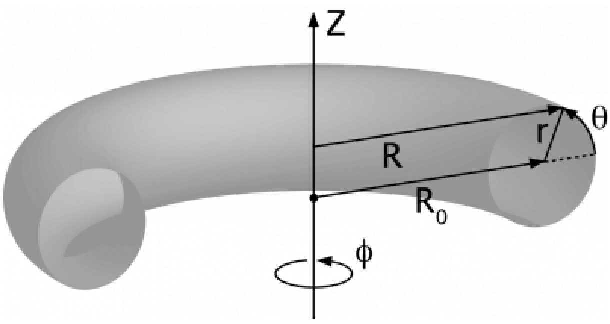

Figure 1: Toroidal coordinates.

Using this two equations, one find that

$$A_{torus} = \sqrt{8\pi^2 R_0 V_{torus}}$$

Substituting, in this last equation, the values given for $R_0 = 1$ m and $V_{torus} = 10\ m^3$, one obtain $A_{torus} = 28.1\ m^2$. The effective area where the wall loading is concentrated[1] is then:

$$A_{1/10} = \frac{A_{torus}}{10} \simeq 2.8 m^2$$

In computing the wall loading, one has to consider the heat flux given by the neutrons, by the α particles and by the losses (mainly radiation and out-flux of particles). The resulting wall load is then:

$$q_{1/10} = \frac{P_n + P_\alpha + P_{in}}{A_{1/10}} = \frac{5.6\ \mathrm{MW} + 1.4\ \mathrm{MW} + 28\ \mathrm{MW}}{2.8\ \mathrm{m}^2} = 12.4\ \mathrm{MW/m}^2$$

This value is higher than the provided limit value, which approximately corresponds to the heat flux supported by the materials we consider today.

[1]This assumption is simplistic since the neutrons and most of the radiation losses coming form the plasma are emitted almost isotropically and cannot be concentrated in such a small region.

Exercise 3 - The effect of impurity contamination on the plasma power balance

Note that the index indicating the charged state of the impurities will be omitted in what follows.

1. a) The bremsstrahlung radiation emitted by carbon impurities is:

$$\begin{aligned} P_{rad}^{C} &= n_C n_e R_C\, V = 0.04\, n_e^2\, R_C\, V \\ &= 0.04 \times 4 \times 10^{40}\ \mathrm{m}^{-6} \times 10^{-34}\ \mathrm{Wm}^3 \times 10\,\mathrm{m}^3 \simeq 1.6\ \mathrm{MW} \end{aligned} \tag{15}$$

   The quasi-neutrality condition of the plasma constrains the DT density. Having a carbon concentration of $n_C = 0.04\ n_e$ it follows that:

$$n_{DT} = n_e - Z_C n_C = n_e(1 - 6 \times 0.04) = 0.76\ n_e$$

   The fractional reduction of the fusion power due to dilution of the fusion fuel from carbon impurity contamination is then:

$$\begin{aligned} F &= 1 - \frac{P_{dilu}^{C}}{P_{norm}} = 1 - \frac{n_D\, n_T \langle\sigma v\rangle_{DT} \Delta E_f}{\frac{1}{4} n_e^2 \langle\sigma v\rangle_{DT} \Delta E_f} = 1 - \frac{\frac{1}{2} n_{DT}\, \frac{1}{2} n_{DT} \langle\sigma v\rangle_{DT} \Delta E_f}{\frac{1}{4} n_e^2 \langle\sigma v\rangle_{DT} \Delta E_f} \\ &= 1 - \frac{(0.76\, n_e)^2}{n_e^2} = 0.422 \simeq 42\% \end{aligned} \tag{16}$$

   b) Proceeding in the same way of the previous question, using tungsten data, we obtain:

$$\begin{aligned} P_{rad}^{W} &= n_W n_e R_W\, V = 10^{-5} n_e^2 R_W\, V \\ &= 10^{-5} \times 4 \times 10^{40}\ \mathrm{m}^{-6} \times 10^{-31}\ \mathrm{W\,m}^3 \times 10\,\mathrm{m}^3 \simeq 400\ \mathrm{kW} \end{aligned} \tag{17}$$

$$n_{DT} = n_e - Z_W n_W = n_e(1 - 50 \times 10^{-5}) = 0.9995\ n_e$$

$$F = 1 - \frac{P_{dilu}^{W}}{P_{norm}} = 1 - \frac{(0.9995\, n_e)^2}{n_e^2} = 10^{-3} \simeq 0.1\% \tag{18}$$

2. One of the aspects that should be considered in a future fusion reactor is that the produced α particles (Helium) have to be exhausted once they have exchanged their fusion energy with the plasma. They would otherwise deteriorate the plasma purity, setting higher constrains on the ignition condition. This implies that the confinement time of α particles has to be low enough to guarantee a small concentration of them. This can be quantified with the parameter ρ^* and the consequent ignition curves shown in the figure of the problem set. A higher value of ρ^* means a higher confinement time of He with respect to the plasma energy confinement time, so a higher concentration of He. As a consequence, the ignition condition corresponds to a smaller region in the parameter space "$n_e\tau_e$ vs. T".