EPFL - EE-406 - Fundamentals of Electrical Circuits and Systems I
Prof. Jean-Philippe Thiran

Lecture notes — Fall 2024

Contents

Chapter 1

Introduction

1.1 Preamble

This document serves as support material for the EPFL course titled *Fundamentals of electrical circuits and systems I* (EE-406).

Most important warning: These lecture notes serve only as a foundation so that we all have the same definitions and main theorems. They *do not* contain explanations or examples, which will be provided in class.

We will closely follow the reference book by Oppenheim and Willsky [1]. If you wish to access more details and a deeper treatment of this subject, you may refer to [2].

This course is one of the important pillars for building your understanding and further exploration of many areas of electrical and electronic engineering and energy.

1.2 Acknowledgements

This document was initially written in English by M. Gastpar and edited by Y. Shkel, to whom I extend my sincere thanks for their help.

1.3 Practical Information, Fall 2024

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Course Web Page: We will use `http://moodle.epfl.ch`

Chapter 2

Signals and Systems

2.1 Signals

2.1.1 Continuous-time versus Discrete-time

As you have seen in your earlier classes, we can model time in two rather different ways, as continuous or as discrete.

A continuous-time signal is a function $x(t)$ that is defined for all times $t \in \mathbb{R}$. By constrast, for a discrete-time signal, we model time as discrete ticks — for example, every second, every nanosecond, or every hour. A discrete-time signal is then a sequence $x[n]$ defined for all integers n. In this class, we will use the square brackets $x[n]$ to denote discrete-time signals.

We should also emphasize that while we often refer to t and n as "time," they may equally well represent space (or any other data indexing).

2.1.2 Continuous-amplitude versus Discrete-amplitude

In most of this class, we will model signals as having *continuous* values. That is, the signal $x(t)$ or $x[n]$ is assumed to be real- or complex-valued.

Of course, in practice, we also sometimes encounter signals that are known to only attain values in a certain fixed, discrete set (for example, the signal might be only either 0 or 1). This is referred to as a discrete-amplitude signal.

2.1.3 Periodic Signals

Periodic signals play a key role for many reasons. From a technological perspective, there are many ways to generate such signals.

A continuous-time signal $x(t)$ is called periodic with period T (where $T > 0$) if for all times t, it satisfies

$$x(t) = x(t+T). \tag{2.1}$$

The *fundamental period* of a periodic signal $x(t)$ is the smallest value of T (where $T > 0$) such that $x(t) = x(t+T)$ holds for all times t.

A discrete-time signal $x[n]$ is called periodic with period N (where N is a positive integer) if for all times n, it satisfies

$$x[n] \quad = \quad x[n+N]. \tag{2.2}$$

The *fundamental period* of a periodic signal $x[n]$ is the smallest positive integer N such that $x[n] = x[n+N]$ holds for all times n.

2.1.4 The Energy of a Signal

For any continuous-time signal $x(t)$, we define its energy by

$$\mathcal{E} \quad = \quad \int_{-\infty}^{\infty} |x(t)|^2 dt. \tag{2.3}$$

If this integral is finite, then $x(t)$ is referred to as an "energy signal."

Similarly, for a discrete-time signal $x[n]$, we define its energy by

$$\mathcal{E} \quad = \quad \sum_{n=-\infty}^{\infty} |x[n]|^2. \tag{2.4}$$

If this infinite sum is finite, then $x[n]$ is referred to as an "energy signal."

2.1.5 The Power of a Signal

Clearly, for every non-zero periodic signal, the energy must be infinite. For those signals, a more interesting quantity would be the *energy per period.* More generally, for any signal, it is interesting to calculate the (average) energy per time unit, which in physics is called the *power.*

For general signals $x(t)$, we formally define the power as

$$\mathcal{P} \quad = \quad \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt. \tag{2.5}$$

If this limit is finite (and strictly larger than zero), then the signal $x(t)$ is referred to as a "power signal."

Similarly, for a discrete-time signal $x[n]$, we define its power by

$$\mathcal{P} \quad = \quad \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2. \tag{2.6}$$

If this limit is finite (and strictly larger than zero), then $x[n]$ is referred to as a "power signal."

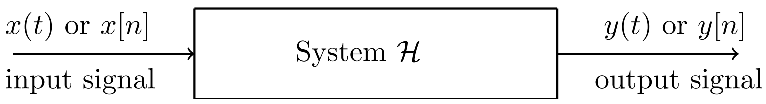


Figure 2.1: A System.

2.2 Systems

A "system" takes a signal as its input and outputs a new signal. We will generally express this as

$$y(t) = \mathcal{H}\{x(t)\} \quad \text{or} \quad y[n] = \mathcal{H}\{x[n]\}, \tag{2.7}$$

where $x(t)$ (or $x[n]$) is the input signal and $y(t)$ (or $y[n]$) is the output signal.

We should alert the reader that both the picture (Figure 2.1) and the notation are slightly misleading: To the untrained eye, it might look as if the particular output sample at time n, which we denote by $y[n]$, is obtained merely from the input sample at that same point in time, denoted by $x[n]$. Obviously, this would not make for an interesting framework! What is meant is that the input is a very long signal $\{x[k]\}_{k=-\infty}^{\infty}$, and the system $\mathcal{H}$ is allowed to use the entire signal in order to produce the output $y[n]$.

2.3 Basic Properties of Systems

The next goal is to *classify* systems. There are many properties that we could isolate. The following six properties are classically considered to be the most powerful.

2.3.1 Linearity

A system $\mathcal{H}\{\cdot\}$ is *linear* if for all signals $x_1(t)$ and $x_2(t)$ and for all constants a_1 and a_2, it is true that

$$\mathcal{H}\{a_1x_1(t) + a_2x_2(t)\} = a_1\mathcal{H}\{x_1(t)\} + a_2\mathcal{H}\{x_2(t)\}, \tag{2.8}$$

or, for the discrete-time case,

$$\mathcal{H}\{a_1x_1[n] + a_2x_2[n]\} = a_1\mathcal{H}\{x_1[n]\} + a_2\mathcal{H}\{x_2[n]\}. \tag{2.9}$$

Otherwise, the system is called *non-linear.*

Examples:

- $y[n] = 2x[n]$ and $y(t) = \int_{-\infty}^{\infty} e^{-|\tau|}x(t-\tau)d\tau$ are linear systems.
- $y(t) = x^2(t)$ and $y[n] = x[n] + 1$ are non-linear systems.

2.3.2 Time-Invariance

A system $\mathcal{H}\{\cdot\}$ is *time-invariant* if for every signal $x(t)$ and for every real number τ, it is true that

if system input $x(t)$ produces system output $y(t)$
then system input $x(t-\tau)$ produces system output $y(t-\tau)$.

In discrete time, it has to hold that for every signal $x[n]$ and every integer number n_0, it is true that

if system input $x[n]$ produces system output $y[n]$
then system input $x[n-n_0]$ produces system output $y[n-n_0]$.

Otherwise, the system is called *time-variant.*

Examples:

- The system $y(t) = x(t) + \frac{1}{2}x(t-\frac{2}{3}) + \frac{1}{3}x(t-3)$ is time-invariant.
- The system $y(t) = x(2t)$ is time-variant.

2.3.3 Memory

A system is called *memoryless* if the current system output only depends on the current system input (for any given point in time).

More formally, a system $\mathcal{H}\{\cdot\}$ is *memoryless* if for all signals $x(t)$ and for every time instant t, the output of the system at time t, $y(t) = \mathcal{H}\{x(t)\}$, is *independent* of the input signal values $x(\tau)$ for all values of $\tau \neq t$.

By the same token, discrete-time systems are called *memoryless* if for all signals $x[n]$ and for every time instant n, the output of the system at time n, $y[n] = \mathcal{H}\{x[n]\}$, is *independent* of the input signal values $x[k]$ for all values of $k \neq n$.

Otherwise, the system is said to *have memory.*

Examples:

- $y(t) = x^2(t)\cos(4t-3)$ is memoryless.
- The system $y(t) = \int_{-\infty}^{\infty} e^{-|\tau|}x(t-\tau)d\tau$ has memory.

It is often interesting to also quantify the amount of memory that a system has. For discrete-time systems, this is easy to do. Consider the system $y[n] = (x[n] - \frac{1}{2}x[n-1])^2\cos(x[n-2])$. Here, a good definition is to say that the system has memory 2 : At any given point in time, the system holds the two previous inputs "in storage."

2.3.4 Invertibility

A system is called *invertible* if distinct inputs lead to distinct outputs.

More formally, a system $\mathcal{H}\{\cdot\}$ is *invertible* if there exists a system $\mathcal{G}\{\cdot\}$ such that for all signals $x(t)$, we have $\mathcal{G}\{\mathcal{H}\{x(t)\}\} = x(t)$. For discrete-time systems, the definition is exactly the same, with t replaced by n.

Examples:

- The system $y(t) = 2x(t)$ is invertible, and the inverse system is $z(t) = \frac{1}{2}y(t)$.
- The system $y[n] = x^2[n]$ is not invertible.

2.3.5 Causality

A system is causal if its output signal only depends on present and past inputs, but not on future inputs.

More formally, a system $\mathcal{H}\{\cdot\}$ is *causal* if for all signals $x(t)$ and for all times τ, the output $y(t) = \mathcal{H}\{x(t)\}$ evaluated at time τ, $y(\tau)$, only depends on the input signal $x(t)$ for times $-\infty < t \leq \tau$.

In discrete time, it has to hold that for all input signals $x[n]$ and all times n_0, the output $y[n] = \mathcal{H}\{x[n]\}$ evaluated at time n_0, $y[n_0]$, only depends on the input signal $x[n]$ for times $-\infty < n \leq n_0$.

Otherwise, the system is called *non-causal.*

Examples:

- The system $y(t) = x(t) + \frac{1}{2}x(t - \frac{2}{3}) + \frac{1}{3}x(t - 3)$ is causal.
- The system $y(t) = x(t) + \frac{1}{2}x(t + \frac{2}{3}) + \frac{1}{3}x(t - 3)$ is not causal.

Anti-Causality

In engineering, it is also convenient to have a notion of *anti-causal* systems. In such a system, for all signals $x(t)$ and for all times τ, the output $y(t) = \mathcal{H}\{x(t)\}$ evaluated at time τ, $y(\tau)$, only depends on the input signal $x(t)$ for times $\tau \leq t < \infty$.

Examples:

- The system $y(t) = x(t) + \frac{1}{2}x(t + \frac{2}{3}) + \frac{1}{3}x(t + 3)$ is anti-causal.
- The system $y(t) = x(t) + \frac{1}{2}x(t + \frac{2}{3}) + \frac{1}{3}x(t - 3)$ is not anti-causal (nor causal).

2.3.6 Stability

To define stability, we first need the notion of a *bounded signal:* A signal $x(t)$ is called *bounded* if there exists a positive constant $B < \infty$ such that $|x(t)| \leq B$ for all times t. For discrete-time systems, the definition is exactly the same, with t replaced by n.

A system $\mathcal{H}\{\cdot\}$ is *stable* if for all *bounded* input signals $x(t)$, the corresponding output signal $y(t) = \mathcal{H}\{x(t)\}$ is also bounded. For discrete-time systems, the definition is exactly the same, with t replaced by n.

Otherwise, the system is called *unstable.*

This notion of stability is often referred to as "BIBO Stability" (bounded-input bounded-output) and will be the only notion of stability discussed in this class.

Examples:

- The system $y(t) = 10^{10}x(t)$ is stable
- The system $y(t) = 1/x(t)$ is not stable.

Chapter 3

Linear Time-Invariant Systems

For the remainder of the class, we will restrict attention to systems that are *linear* and *time-invariant,* often referred to as LTI systems.

3.1 The Impulse Response (discrete-time)

In this section, we derive the fact that LTI systems can be characterized in a convenient way via their so-called *impulse response.* Let us first consider a discrete-time LTI system with input $x[n]$. In full generality, the output can be expressed as $y[n] = \mathcal{H}\{x[n]\}$. Now we want to exploit the fact that the system is assumed to be both linear and time-invariant in order to simplify this. We start with a trick. For each integer k, we introduce the function

$$\delta_k[n] \quad = \quad \begin{cases} 1, & \text{for } n = k, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

Then, if we denote $x_k = x[k]$, we can express the signal $x[n]$ equivalently as

$$x[n] \quad = \quad \sum_{k=-\infty}^{\infty} x_k \delta_k[n]. \tag{3.2}$$

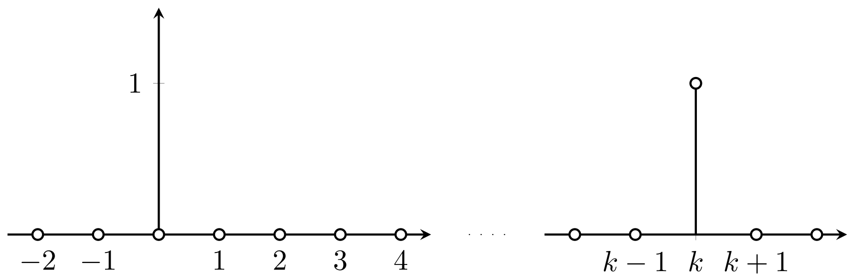


Figure 3.1: The Kronecker-delta function $\delta_k[n]$.

Now, we can exploit the *linearity* of the system, as follows:

$$y[n] = \mathcal{H}\left\{\sum_{k=-\infty}^{\infty} x_k\delta_k[n]\right\} \tag{3.3}$$

$$\stackrel{(*)}{=} \sum_{k=-\infty}^{\infty} x_k\mathcal{H}\left\{\delta_k[n]\right\}, \tag{3.4}$$

where step $(*)$ is due to the linearity. For the next step, it is better to switch to a more standard notation. Let us define the so-called *Kronecker-delta function*:

$$\delta[n] = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

With this, we can express $\delta_k[n] = \delta[n-k]$. Therefore, we can equivalently express Equation (3.2) as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]. \tag{3.6}$$

and Equation (3.4) as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\mathcal{H}\left\{\delta[n-k]\right\}. \tag{3.7}$$

To proceed, we next define the signal

$$h[n] = \mathcal{H}\left\{\delta[n]\right\}, \tag{3.8}$$

which is simply the system response when the input is the Kronecker-delta function $\delta[n]$. Therefore, the signal $h[n]$ is called the *impulse response* of the system $\mathcal{H}\{\cdot\}$. Next, we use the fact that the system is *time-invariant.* Specifically, time invariance means that the system reponse to a shifted Kronecker-delta function $\delta[n-k]$ is precisely

$$\mathcal{H}\left\{\delta[n-k]\right\} = h[n-k]. \tag{3.9}$$

This means that we can equivalently characterize the system output signal as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \tag{3.10}$$

The fundamental upshot is that *any* LTI system is uniquely characterized by its impulse response $h[n]$. Moreover, given the impulse response, it is easy to determine the system output for any input signal $x[n]$ using the formula given in Equation (3.10). This formula is important enough to have its own name: It is referred to as the *convolution sum* (or the convolution operation).

3.2 The Impulse Response (continuous-time)

Now, let us turn to continuous-time systems. The key step is to express the input signal in the form given in Equation (3.6). That is, we are looking for a "function" $\delta(t)$ that satisfies, for any signal $x(t)$ that is continuous at t, the identity $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$. Unfortunately, this is not quite as straightforward as in the discrete-time case. For example, defining $\delta(t) = 1$ for $t = 0$, and $\delta(t) = 0$ otherwise (like in the discrete-time case), clearly does not satisfy the desired identity. In fact, in a similar fashion, it is easy to show that no "standard" function can hope to satisfy the desired identity.

By generalizing the notion of functions, one can indeed define a mathematical object $\delta(t)$, called the *(Dirac) delta function,* satisfying

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \tag{3.11}$$

for all t at which $x(t)$ is continuous. Regarding $\delta(t)$ itself, we can say that it has to be equal to zero whenever $t \neq 0$. At $t = 0$, its value is "infinite" in the sense that Equation (3.11) ends up holding.

Using the Dirac delta function, we can again proceed as follows:

$$y(t) = \mathcal{H}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\} \tag{3.12}$$

$$\stackrel{(*)}{=} \int_{-\infty}^{\infty} x(\tau)\mathcal{H}\left\{\delta(t-\tau)\right\} d\tau, \tag{3.13}$$

where step $(*)$ is again due to the linearity. To proceed, we assume that it is fine to define the signal

$$h(t) = \mathcal{H}\left\{\delta(t)\right\}, \tag{3.14}$$

which is simply the system response when the input is $\delta(t)$. Next, we use the fact that the system is *time-invariant.* Specifically, time invariance means that the system's reponse to a shifted $\delta(t-\tau)$ is precisely

$$\mathcal{H}\left\{\delta(t-\tau)\right\} = h(t-\tau). \tag{3.15}$$

This means that we can equivalently characterize the system output signal as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau. \tag{3.16}$$

This is the powerful result that we are after: It again shows that any LTI system can be characterized by its *impulse response* $h(t)$, and that for any input signal $x(t)$, the corresponding system output signal $y(t)$ can be found using the formula given in Equation (3.16). This formula is referred to as the *convolution integral* (or the convolution operation).

The remaining problem is to make the steps leading up to the fundamental Equation (3.16) precise. An important observation is that the delta function $\delta(t)$ is *no longer* *present* in this final expression. In order to fully accomplish the derivation, a mathematical theory has been developed under the heading of "generalized functions" or "distributions." We encourage the mathematically inclined reader to study this in more detail. In class, we present a more direct argument using the mathematical concept of limits.

3.3 The Convolution Operation

As we have seen, LTI systems are fully described by their impulse response: The output signal is simply given by the convolution of the input signal with the impulse response. In this section, we explore some aspects of convolution. In fact, convolution is important enough to have its own symbol. In many texts, it is denoted as

$$(x * h)[n] \quad = \quad \sum_{k=-\infty}^{\infty} x[k]h[n-k], \tag{3.17}$$

or

$$(x * h)(t) \quad = \quad \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau, \tag{3.18}$$

and in this class, we will follow this convention. However, in other texts, you will also see convolution denoted as $x[n] * h[n]$ or $x(t) * y(t)$.

3.3.1 Convolution with the Delta Function

Let us now consider the important special case $(x * \delta)[n]$, i.e., the convolution of a signal $x[n]$ with the Kronecker-delta function $\delta[n]$. We find

$$(x * \delta)[n] \quad = \quad \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \quad = \quad x[n], \tag{3.19}$$

since the function $\delta[n-k]$ is zero except if $k = n$, when it is one. That is, the convolution of any signal with the delta function simply returns back that signal.

In continuous time, we can similarly study $(x * \delta)(t)$, i.e., the convolution of a signal $x(t)$ with the Dirac delta function. We find

$$(x * \delta)(t) \quad = \quad \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \quad = \quad x(t), \tag{3.20}$$

which is of course simply Equation (3.11) that we have studied earlier (and thus, it comes with the same limitations; in particular, the last equality holds whenever $x(t)$ is continuous). That is, the convolution of any signal with the delta function simply returns back that signal.

3.3.2 The Commutative Property

A first interesting question is whether we can *swap* the order of the two signals, i.e., whether we have that $(x * h)[n] = (h * x)[n]$. Let us write out:

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \stackrel{(a)}{=} \sum_{\tilde{k}=\infty}^{-\infty} x[n-\tilde{k}]h[\tilde{k}] \quad (3.21)$$

$$= \sum_{\tilde{k}=-\infty}^{\infty} x[n-\tilde{k}]h[\tilde{k}] = (h * x)[n], \quad (3.22)$$

where for the summation variable substitution in step (a), we have defined $\tilde{k} = n - k$ (and we recall that n is a constant). Hence, the answer is positive: we can swap the order.

For the continuous-time case, the argument is essentially the same, as follows:

$$(x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \stackrel{(b)}{=} \int_{\infty}^{-\infty} x(t-\tilde{\tau})h(\tilde{\tau})(-d\tilde{\tau}) \quad (3.23)$$

$$= -\int_{-\infty}^{\infty} x(t-\tilde{\tau})h(\tilde{\tau})(-d\tilde{\tau}) = \int_{-\infty}^{\infty} x(t-\tilde{\tau})h(\tilde{\tau})d\tilde{\tau} \quad (3.24)$$

$$= (h * x)(t), \quad (3.25)$$

where for the integration variable substitution in step (b), we have defined $\tilde{\tau} = t - \tau$ (and we recall that t is a constant). Hence, again, the answer is positive: we can swap the order.

3.3.3 The Distributive Property

The next question is whether we have that

$$(x * (h_1 + h_2))[n] = (x * h_1)[n] + (x * h_2)[n] \quad (3.26)$$

or, in continuous time,

$$(x * (h_1 + h_2))(t) = (x * h_1)(t) + (x * h_2)(t). \quad (3.27)$$

For both cases, the answer is positive and can be verified in a straightforward way, as we will show in class.

3.3.4 The Associative Property

Finally, we consider whether we have that

$$((x * h_1) * h_2)[n] = (x * (h_1 * h_2))[n], \quad (3.28)$$

or, in continuous time,

$$((x * h_1) * h_2)(t) = (x * (h_1 * h_2))(t). \quad (3.29)$$

That is, the question is whether multiple convolutions can be executed in an arbitrary order. Here, it turns out that the answer is somewhat more subtle. The property holds if all three involved signals are absolutely summable (*i.e.,* $\sum_{n\in\mathbb{Z}}|x[n]| < \infty, \sum_{n\in\mathbb{Z}}|h_1[n]| < \infty, \sum_{n\in\mathbb{Z}}|h_2[n]| < \infty$) or, in the continuous-time case, absolutely integrable. This is a sufficient condition for associativity to hold, but it is not necessary — associativity may also hold in more general cases. But there are also simple cases where associativity does not hold, see Appendix 3.A for one such example.

3.3.5 Techniques to Evaluate Convolution Sums and Integrals

There are several techniques to evaluate convolution. For a thorough understanding of LTI systems, it is important to develop a certain intuition for several of them. We will do this in class and in the Problem Sets. The techniques are:

- "graphical" convolution (also called "flip-and-drag")
- analytical, i.e., simply by solving the integral (or the sum) in closed form
- numerical (i.e., using numerical software).

3.4 Composition of LTI Systems

A further reason why LTI systems are such powerful models is because when multiple LTI systems are combined in several natural ways, the new overall system is again an LTI system. The most common compositions will be discussed in the sequel.

3.4.1 Parallel

For two systems $\mathcal{H}_1$ and $\mathcal{H}_2$ in parallel composition (Figure 3.2), we can express the overall system $\mathcal{G}$ simply as

$$y(t) \quad = \quad \mathcal{G}\{x(t)\} = \mathcal{H}_1\{x(t)\} + \mathcal{H}_2\{x(t)\}. \tag{3.30}$$

If both $\mathcal{H}_1$ and $\mathcal{H}_2$ are LTI systems, then $\mathcal{G}$ is also an LTI system. Moreover, the impulse response $g(t)$ (or $g[n]$) of the system $\mathcal{G}$ is simply given by

$$g(t) = h_1(t) + h_2(t) \quad \text{or} \quad g[n] = h_1[n] + h_2[n]. \tag{3.31}$$

3.4.2 Series

For two systems $\mathcal{H}_1$ and $\mathcal{H}_2$ in series composition (Figure 3.3), we can express the overall system $\mathcal{G}$ simply as

$$y(t) \quad = \quad \mathcal{G}\{x(t)\} = \mathcal{H}_2\{\mathcal{H}_1\{x(t)\}\}. \tag{3.32}$$

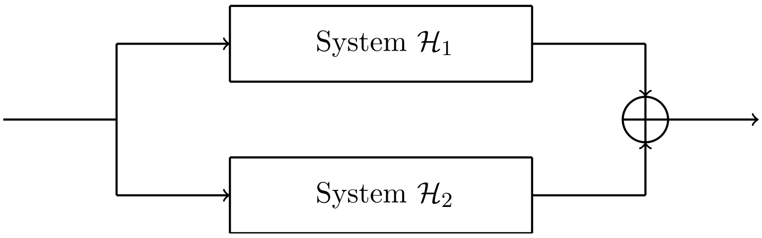


Figure 3.2: Parallel Composition of two systems.

If both $\mathcal{H}_1$ and $\mathcal{H}_2$ are LTI systems, then $\mathcal{G}$ is also an LTI system. Moreover, if both systems are stable, the impulse response $g(t)$ (or $g[n]$) of the system $\mathcal{G}$ is simply given by

$$g(t) = \int_{-\infty}^{\infty} h_1(\tau)h_2(t-\tau)d\tau, \quad \text{or} \quad g[n] = \sum_{k=-\infty}^{\infty} h_1[k]h_2[n-k]. \tag{3.33}$$

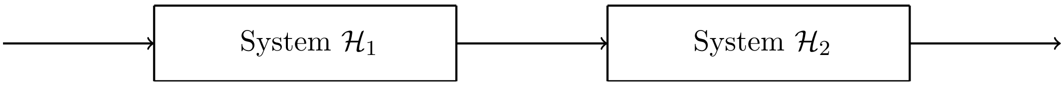


Figure 3.3: Series Composition of two systems.

3.5 Properties of LTI Systems

We are studying systems that are both linear and time-invariant. In this section, we consider the remaining three fundamental system properties. As we have seen, LTI systems are fully described by their impulse response $h(t)$ (or $h[n]$). Hence, our discussion will connect the fundamental system properties to the particular shape of the impulse response.

3.5.1 Memory

An LTI system is memoryless if and only if, for some constant a, we have

$$y(t) = ax(t) \quad \text{or} \quad y[n] = ax[n]. \tag{3.34}$$

What does this imply for the impulse response? Let us first take up the discrete-time case. Here, using the definition of the Kronecker-delta function $\delta[n]$, we can see that we must have

$$h[n] \;=\; a\delta[n]. \tag{3.35}$$

The situation is a little more involved in continous time. But using the concept of the delta function $\delta(t)$ as discussed above, we can see that for a memoryless LTI system, we must have

$$h(t) = a\delta(t). \tag{3.36}$$

3.5.2 Invertibility

An LTI system with impulse response $h(t)$ is invertible if and only if there exists a function $g(t)$ such that $(g * h)(t) = \delta(t)$. For discrete-time systems, the definition is exactly the same, with t replaced by n.

3.5.3 Causality

An LTI system is causal if and only if the impulse response function is identically zero for negative lags: $h(t) = 0$ for $t < 0$ (or $h[n] = 0$ for $n < 0$).

3.5.4 Stability

An LTI system is stable if and only if the impulse response is absolutely integrable (or summable), i.e., if and only if

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty \quad \text{or} \quad \sum_{k=-\infty}^{\infty} |h[k]| < \infty. \tag{3.37}$$

A proof of this result will be developed in class.

3.6 Systems Modeled by Differential Equations

Many systems are naturally described by differential equations (for example, using the laws of physics). A fundamental insight is that any system that is described by a *linear* differential equation (with constant coefficients) constitutes an LTI system. To be more precise, we also have to add that this is only true if the initial conditions for the system are all zero. (If they are not zero, the system is still almost linear, and many of the techniques that we will see in this class can be modified to be applied to these systems.)

We will start our discussion with the consideration of discrete-time systems. In this case, differential equations are usually referred to as *difference equations.* Here is a simple example of such a *linear difference equation with constant coefficients:*

$$y[n] + 3y[n-1] = x[n] - \frac{1}{2}x[n-1]. \tag{3.38}$$

Our goal is to argue that indeed, this describes an LTI system. Unfortunately, there is a small hiccup that we need to discuss first. In particular, let us suppose that the input signal $x[n]$ is zero for all $n < 0$. Then, for $n = 0$, we obtain the relationship $y[0] + 3y[-1] = x[0]$, and we see that we also explicitly need to specify $y[-1]$. A tempting choice is to simply set

$y[-1] = 0$. In this case, it is easy to verify that the resulting system is indeed both linear and time-invariant. By the same reasoning, it can be seen that *any* system described by a linear difference equation with constant coefficients and where all initial conditions are set to zero describes an LTI system. For continuous-time systems, the analogous consideration concerns *differential equations* like the ones you have seen in your Analysis class. As you remember, to solve those equations, it was necessary to specify *initial conditions.* Again, if the differential equation is a *linear differential equation with constant coefficients* and the initial conditions are set to "rest" (meaning that all signals and derivatives that appear in the equation are initially set to zero), then this describes an LTI system.

Now that we know that Equation (3.38) describes an LTI system, it is natural to ask what its *impulse response* is. This can be done explicitly by solving the difference (or differential) equation. However, in this class, we will see a number of much more powerful and efficient tools to solve this problem.

Appendix 3.A Exchanging Integration and Summation Order

For multiple stacked sums (or integrals), it is often convenient to arrange the summation (or integration) order as one wishes. For example, in Subsection 3.3.4, we observed that "in many cases," convolution is associative, and we stated a sufficient condition.[1] In this appendix, we give a simple explicit counterexample, that is, a case where one *cannot* swap the summation order. Consider the following three signals, defined for $-\infty < n < \infty$:

$$d[n] = \begin{cases} 1, & n = 0 \\ -1, & n = 1 \\ 0, & \text{otherwise.} \end{cases} \tag{3.39}$$

$$e[n] = 1, \forall n \tag{3.40}$$

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{3.41}$$

Let us compute both

$$(u * (d * e))[n] \quad \text{and} \quad ((u * d) * e)[n], \tag{3.42}$$

in the specific order as indiciated by the brackets. For the left version, we first calculate

$$(d * e)[i] = \sum_{m=-\infty}^{\infty} d[m]e[i - m] = \sum_{m=-\infty}^{\infty} d[m] = 0 \tag{3.43}$$

[1] A more general sufficient condition to guarantee that the integration/summation order can be swapped is the so-called *Fubini* theorem, which can be found in any textbook on Analysis, see for example [3, Theorems 8.6 and 8.8], but note that no deeper understanding of these theorems is required for our class.

hence $(u * (d * e))[n] = 0$ for all n. On the other hand, for the version on the right, we first calcuate

$$(u * d)[l] = \sum_{k=-\infty}^{\infty} u[k]d[l-k] = \sum_{k=0}^{\infty} d[l-k] \tag{3.44}$$

$$= \begin{cases} 1 & l = 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.45}$$

hence $((u * d) * e)[n] = e[n] = 1$ for all n. As a final remark, recall the simple sufficient condition stated in Subsection 3.3.4: If all involved signals are absolutely summable (or integrable), then convolution is associative. This condition is clearly not satisfied in the above example: Neither $e[n]$ nor $u[n]$ are absolutely summable. Hence, the above example does not contradict our sufficient condition. (However, let us again stress that the condition of absolute summability is *not* a necessary condition for swapping summation order.)

Chapter 4

Fourier Methods for Stable LTI Systems

In the previous chapter, we have seen that an output of an LTI system for a given input signal could be computed via convolution between the input signal and the impulse response of the LTI system. This characterization of LTI systems is based on representing signals as linear combinations of shifted impulses. In this and the following chapters, we explore an alternative representation for signals and LTI systems as linear combinations of *complex exponentials*. As we will see, this new representation will gives us new and powerful tools for the analysis of LTI systems.

The next task is to discuss *how* we can represent signals as linear combinations of complex exponentials. These techniques are due to Joseph Fourier (1768-1830), and in his honor, are called *Fourier transform* and *Fourier series*.

A more subtle question concerns the existence of such representations. Although many signals of interest can be expressed as linear combinations of complex exponentials, there are exceptions and technicalities. Many of these were resolved by other mathematicians later on, including Cauchy and Dirichlet. You see a more detailed account of these issues in your Analysis III-IV classes.

A good answer for the purpose of engineering is that Fourier representations are well defined for

- Periodic signals

- Absolutely integrable signals, i.e., signals for which $\int_{-\infty}^{\infty} |x(t)|dt$ is finite. This class of functions is often referred to as L^1.

- Finite energy signals, i.e., signals for which $\int_{-\infty}^{\infty} |x(t)|^2 dt$ is finite. This class of functions is often referred to as L^2.

4.1 Continuous-time Signals

4.1.1 The Fourier Transform

For a signal $x(t)$, we define the *Fourier transform* as

$$X(\omega) \;=\; \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt, \tag{4.1}$$

if this integral converges. In your Analysis III-IV courses, you will study this question in more detail. A sufficient condition for convergence is that the signal $x(t)$ is absolutely integrable, *i.e.,* we have $\int_{-\infty}^{\infty} |x(t)|dt < \infty$. The *inverse Fourier transform* is defined as

$$x(t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega, \tag{4.2}$$

again if this integral converges.

$X(\omega)$ is called the *spectrum* of the signal $x(t)$. Intuitively, the (complex) number $X(\omega)$ determines "how much" of every frequency ω is present in the signal $x(t)$. When we talk about the Fourier transform, $X(\omega)$, of a signal, we say that we consider the signal in the *frequency-domain* rather than $x(t)$ which is the representation of the signal in the *time-domain.*

A key question is the following one : If we start with a signal $x(t)$, apply the Fourier transform to obtain $X(\omega)$, and then apply the inverse Fourier transform, do we get back the same signal $x(t)$? (In other words, is the Fourier transform invertible?) For the purpose of our class, the answer is *yes* — we will always use the Fourier transform in this sense. However, strictly speaking, this invertibility only applies to well-behaved signals and has a good number of footnotes and restrictions, some of which you will encounter in your Analysis III-IV courses. We include a few brief comments in Section 4.3.

4.1.2 Integrating a Complex Exponential

For the remainder of this class, one simple integral (that you know very well) will play a key role. To be on the safe side, we include it here. Namely, for any complex number c, we have

$$\int_0^{\infty} e^{-ct}dt = \frac{1}{c},$$

if $Re\{c\} > 0$. Otherwise, the integral does not converge. Note that by a change of integration variable $\tau = -t$, this also implies

$$\int_{-\infty}^{0} e^{c\tau}d\tau = \int_{\infty}^{0} e^{-ct}(-dt) = \int_0^{\infty} e^{-ct}dt = \frac{1}{c},$$

again if $Re\{c\} > 0$. (Again, otherwise, the integral does not converge.)

It is assumed that you are fully familiar with the case where c is a real number. To derive the case when c is complex-valued, you may proceed as follows: Write $c = c_R + jc_I$ and observe

$$\begin{aligned} \int_0^\infty e^{-ct}dt &= \int_0^\infty e^{-c_R t}\cos(c_I t)dt - j\int_0^\infty e^{-c_R t}\sin(c_I t)dt \\ &= \frac{c_R}{c_R^2 + c_I^2} - j\frac{c_I}{c_R^2 + c_I^2}, \end{aligned}$$

where the second expression involves only standard real-valued integrals that can be looked up in your favorite integral table. Combining,

$$\frac{c_R}{c_R^2 + c_I^2} - j\frac{c_I}{c_R^2 + c_I^2} = \frac{c^*}{|c|^2} = \frac{1}{c}.$$

4.1.3 Properties of the Fourier Transform

Our keen interest will now be devoted to exploring the various properties of the Fourier transform. Tables with many of these properties are given at the end of the chapter in Appendices 4.A and 4.B.

At this point, we consider only one very special function, often referred to as the "box function:"

$$b(t) = \begin{cases} 1, & |t| \leq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

Its Fourier transform is found by direct evaluation:

$$B(\omega) = \int_{-1}^{1} e^{-j\omega t}dt = \frac{1}{j\omega}e^{j\omega} - \frac{1}{j\omega}e^{-j\omega} = \frac{2\sin(\omega)}{\omega}, \tag{4.4}$$

where for the last step, we have used the identity $\sin(x) = \frac{1}{2j}(e^{jx} - e^{-jx})$.

Since this last function appears frequently, engineers have introduced a special symbol:

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}, \tag{4.5}$$

called the "sinc-function" (pronounced essentially like "sink-function"). Using this, we can express Equation (4.4) equivalently as

$$B(\omega) = 2\,\mathrm{sinc}(\omega/\pi). \tag{4.6}$$

4.1.4 LTI Systems and the Fourier Transform

The most fundamental insight of our class is that for any LTI system, we can express the input-output relationship as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau, \tag{4.7}$$

which was Equation (3.16). But then,

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right) e^{-j\omega t}dt. \end{aligned} \tag{4.8}$$

In this chapter, we assume that the system is stable, which is the same as assuming that $h(t)$ is absolutely integrable. Moreover, we assume that the Fourier transform of $x(t)$ exists. Then, we may swap integration order to obtain

$$Y(\omega) = \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt\right) d\tau. \tag{4.9}$$

Let us define

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt \tag{4.10}$$

to be the Fourier transform of the impulse response $h(t)$. We call $H(\omega)$ the *frequency response* of the system and will look at it in more detail in the next chapter. For now, the inner integral in (4.9) can be solved for example using the "shift in time" property, leading to $e^{-j\omega\tau}H(\omega)$. Thus, we are left with

$$\begin{aligned} Y(\omega) &= H(\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \\ &= H(\omega)X(\omega). \end{aligned} \tag{4.11}$$

That is, if we are analyzing LTI systems, then it is an excellent idea to consider signals in the frequency-domain rather than the time-domain: In the time domain, the system operation is given by the convolution, which, as we saw, is a little tedious to evaluate in many cases. By contrast, in the frequency domain, the system operation is simply characterized by (point-wise) multiplication, a much simpler operation!

This insight is also often referred to as the *convolution property* of the Fourier transform, stating that

$$(h * x)(t) \quad \circ\!\!-\!\!\bullet \quad H(\omega)X(\omega), \tag{4.12}$$

where we use the shorthand symbol $\circ\!\!-\!\!\bullet$ to mean that on the open circle side is the time-domain representation and on the closed bullet side is the corresponding frequency domain representation.

4.1.5 LTI Systems and the Fourier Transform: Differential Equations

In many cases, continuous-time LTI systems are conveniently described by differential equations. For those, one could first solve the differential equation assuming that the input is the delta function, thus finding the impulse response $h(t)$. However, there is a much easier approach in this case. To see this, we first observe that if a certain signal $x(t)$ has Fourier transform $X(\omega)$, then the signal $\frac{d}{dt}x(t)$ must have Fourier transform $j\omega X(\omega)$. In our notation,

$$\frac{d}{dt}x(t) \quad \circ\!\!-\!\!\bullet \quad j\omega X(\omega). \tag{4.13}$$

But then, we observe that if two signals $x(t)$ and $y(t)$ satisfy

$$y(t) + 3\frac{d}{dt}y(t) + 2\frac{d^2}{dt^2}y(t) \quad = \quad x(t) + \frac{1}{2}\frac{d}{dt}x(t), \tag{4.14}$$

then their Fourier transforms $X(\omega)$ and $Y(\omega)$ must satisfy

$$Y(\omega) + 3j\omega Y(\omega) + 2(j\omega)^2 Y(\omega) \quad = \quad X(\omega) + \frac{1}{2}j\omega X(\omega). \tag{4.15}$$

First, we trivially rearrange this equation to

$$Y(\omega)\left(1 + 3j\omega + 2(j\omega)^2\right) \quad = \quad X(\omega)\left(1 + \frac{1}{2}j\omega\right). \tag{4.16}$$

To see why this insight is highly valuable, we have to recall from Equation (4.11) that $Y(\omega) = H(\omega)X(\omega)$, which implies that $H(\omega) = Y(\omega)/X(\omega)$. That is, we find

$$H(\omega) \quad = \quad \frac{1 + \frac{1}{2}j\omega}{1 + 3j\omega + 2(j\omega)^2}. \tag{4.17}$$

In other words, we found the frequency response of the system in a couple of very easy steps, without solving any differential equations! (If needed, we can now take the inverse Fourier transform of $H(\omega)$ to recover the impulse response $h(t)$.)

We should also note that this approach does not just work for the example differential equation above; it should be clear that it works for *any* constant-coefficient linear differential equation whatsoever.

4.1.6 Delta Functions and the Fourier Transform

Fourier methods are useful even in cases where the Fourier transform (4.1) does not converge. For example, let us pretend that a certain signal $z(t)$ has Fourier transform given by

$$Z(\omega) \quad = \quad 2\pi\delta(\omega - \omega_0), \tag{4.18}$$

for a certain (fixed) value of ω_0. To find the signal $z(t)$, we have to apply the inverse Fourier transform, Formula (4.2),

$$z(t) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}, \tag{4.19}$$

where for the last step, we have used the fundamental property of the Dirac-delta function in Equation (3.11) and the fact that $e^{j\omega t}$ is a continuous function of ω. In particular, we also point out that for the special choice $\omega_0 = 0$, we find that the corresponding time domain signal is $z(t) = 1$ for all $t, -\infty < t < \infty$.

The key observation is that while the Fourier transform $Z(\omega)$ may be a tricky object to deal with, the corresponding time-domain signal is a perfectly well-behaved function, making the overall argument very useful in practice.

By analogy, we can also consider the time-domain signal $y(t) = \delta(t - t_0)$, which is a Dirac-delta at time t_0. Its Fourier transform is found to be

$$Y(\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0}. \tag{4.20}$$

As we will see, this "generalized" Fourier transform turns out to be very useful in practice. Nevertheless, we should end this discussion on a cautionary note: These arguments must be used with some care. To be precise, correctness must be argued on a case-by-case basis.

4.2 Discrete-time Signals

4.2.1 The Discrete-time Fourier Transform

For a discrete-time signal $x[n]$, we define the *discrete-time Fourier transform* as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \tag{4.21}$$

if this sum converges.

The *inverse discrete-time Fourier transform* is defined as

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega. \tag{4.22}$$

Again, there is a somewhat more subtle question concerning the invertibility of the discrete-time Fourier transform, but for the purpose of our class, and for the (well-behaved) signals that are of interest to us, invertibility will always hold, making the discrete-time Fourier transform an important tool.

4.2.2 Properties of the Discrete-time Fourier Transform

A simple (but important) fact is that the discrete-time Fourier transform is 2π-periodic. That is, for every integer k, we have

$$X(\omega + k2\pi) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega + k2\pi)n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \underbrace{e^{-j2\pi kn}}_{=1} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = X(\omega).$$

Tables with many of these properties are given at the end of the chapter in Appendices 4.C and 4.D.

4.2.3 LTI Systems and the Discrete-time Fourier Transform

The most fundamental insight of our class is that for any discrete-time LTI system, we can express the input-output relationship as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \tag{4.23}$$

which was Equation (3.10). But then,

$$\begin{aligned} Y(\omega) &= \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]h[n-k] \right) e^{-j\omega n}. \end{aligned} \tag{4.24}$$

In this chapter, we assume that the system is stable, hence, that $h[n]$ is absolutely summable. Moreover, we assume that the discrete-time Fourier transform of $x[n]$ exists. Then, we may swap summation order to obtain

$$Y(\omega) = \sum_{k=-\infty}^{\infty} x[k] \left(\sum_{n=-\infty}^{\infty} h[n-k]e^{-j\omega n} \right). \tag{4.25}$$

The inner sum can be solved for example using the "shift in time" property, leading to $e^{-j\omega k}H(\omega)$ (where again, the frequency response $H(\omega)$ is the discrete-time Fourier transform of the impulse response). Thus, we are left with

$$\begin{aligned} Y(\omega) &= H(\omega) \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k} \\ &= H(\omega)X(\omega). \end{aligned} \tag{4.26}$$

Again, we see that if we switch from time-domain to frequency domain, the convolution operation becomes a simple point-wise multiplication. Exactly like in the continuous-time case, this insight is also often referred to as the *convolution property* of the discrete-time Fourier transform, stating that

$$(h * x)[n] \quad \circ\!\!-\!\!\bullet \quad H(\omega)X(\omega). \tag{4.27}$$

4.2.4 LTI Systems and the Discrete-time Fourier Transform: Difference Equations

In many cases, discrete-LTI systems are conveniently described by difference equations. For those, one could first solve the difference equation assuming that the input is the Kronecker-delta function, thus finding the impulse response $h[n]$. However, there is a much easier approach in this case. To see this, we first observe that if a certain signal $x[n]$ has discrete-time Fourier transform $X(\omega)$, then the signal $x[n-1]$ must have Fourier transform $e^{-j\omega}X(\omega)$. In our notation,

$$x[n-1] \quad \circ\!\!-\!\!\bullet \quad e^{-j\omega}X(\omega). \tag{4.28}$$

But then, we observe that if two signals $x[n]$ and $y[n]$ satisfy

$$y[n] - \frac{1}{2}y[n-1] \;=\; x[n] + \frac{2}{3}x[n-1] - \frac{3}{5}x[n-2], \tag{4.29}$$

then their discrete-time Fourier transforms $X(\omega)$ and $Y(\omega)$ must satisfy

$$Y(\omega) - \frac{1}{2}e^{-j\omega}Y(\omega) \;=\; X(\omega) + \frac{2}{3}e^{-j\omega}X(\omega) - \frac{3}{5}e^{-j2\omega}X(\omega). \tag{4.30}$$

First, we trivially rearrange this equation to

$$Y(\omega)\left(1 - \frac{1}{2}e^{-j\omega}\right) \;=\; X(\omega)\left(1 + \frac{2}{3}e^{-j\omega} - \frac{3}{5}e^{-j2\omega}\right). \tag{4.31}$$

To see why this insight is highly valuable, we have to recall from Equation (4.26) that $Y(\omega) = H(\omega)X(\omega)$, which implies that $H(\omega) = Y(\omega)/X(\omega)$. That is, we find

$$H(\omega) \;=\; \frac{1 + \frac{2}{3}e^{-j\omega} - \frac{3}{5}e^{-j2\omega}}{1 - \frac{1}{2}e^{-j\omega}}. \tag{4.32}$$

In other words, we found the frequency response of the system in a couple of very easy steps, without solving any difference equations! (If needed, we can now take the inverse discrete-time Fourier transform of $H(\omega)$ to recover the impulse response $h[n]$.)

We should also note that this approach does not just work for the example difference equation above; it should be clear that it works for *any* constant-coefficient linear difference equation whatsoever.

4.2.5 Delta Functions and the Discrete-time Fourier Transform

The discrete-time Fourier transform is useful even in cases where the expression in Formula (4.21) does not converge. For example, consider a signal $z[n]$ whose discrete-time Fourier transform $Z(\omega)$ has a Dirac-delta of amplitude 2π at frequency ω_0, where $|\omega_0| < \pi$. At this point, it is important to recall that $Z(\omega)$ *must* be 2π-periodic. In other words, if there is a Dirac-delta at frequency ω_0, then there must be Dirac-deltas at all frequencies $\omega_0 + 2\pi\ell$, for all integers ℓ. One can express such a "train" of Dirac-deltas as

$$Z(\omega) \;=\; \sum_{\ell=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi\ell). \tag{4.33}$$

We can now use the inverse discrete-time Fourier transform (Formula (4.22)) to determine the corresponding time-domain signal $z[n]$ as

$$z[n] \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\ell=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi\ell)\right)e^{j\omega n}d\omega. \tag{4.34}$$

In this doubly infinite sum, only a single Dirac-delta function falls into the interval from $-\pi$ to π (namely, the one where $\ell = 0$). But since the integral over ω only concerns this interval, we find

$$z[n] = \int_{-\pi}^{\pi} \delta(\omega - \omega_0) e^{j\omega n} d\omega = e^{j\omega_0 n} \tag{4.35}$$

where for the last step, we have used the fundamental property of the Dirac-delta function in Equation (3.11) and the fact that $e^{j\omega n}$ is a continuous function of ω. An interesting special case is the choice $\omega_0 = 0$. Now, the Dirac-delta in $Z(\omega)$ is located at frequency zero (plus multiples of 2π), and the corresponding time-domain signal is the all-ones sequence $z[n] = 1$.

4.3 Convergence Issues

As you see in more detail in your Analysis III-IV classes, a fundamental question is the convergence of the Fourier integrals. One way in which we can ask this question is: If we start with a signal $x(t)$ and calculate the Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt, \tag{4.36}$$

and then invert this to obtain

$$\tilde{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega, \tag{4.37}$$

do we really have $x(t) = \tilde{x}(t)$? So far, we have just pretended that this is the case. But there are some subtleties.

A first case where one can make general statements concerns all those signals $x(t)$ that are absolutely integrable (i.e., $\int_{-\infty}^{\infty} |x(t)| dt < \infty$). For those signals, we have $x(t) = \tilde{x}(t)$ for all values of t *except* for those t for which $x(t)$ is not continuous. At these points of discontinuity, the interesting and important *Gibbs phenomenon* occurs. (If you are curious, enter this search term into your internet search engine of preference and you will find a plethora of documents and animations.)

A second case concerns the finite-energy signals, which are those signals $x(t)$ for which $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$. For these signals, a successful perspective comes in the shape of a mathematical concept called *Hilbert space.*

Appendix 4.A Continuous-Time Fourier Transform : Properties

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \quad \longleftrightarrow \quad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Property	**Signal**	**Fourier transform**
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(\omega) + \beta Y(\omega)$
Shift in time	$x(t - t_0)$	$e^{-jt_0\omega}X(\omega)$
Shift in frequency	$e^{j\omega_0 t}x(t)$	$X(\omega - \omega_0)$
Scaling in time and frequency	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Time reversal	$x(-t)$	$X(-\omega)$
Differentiation in time	$\frac{d^n}{dt^n}x(t)$	$(j\omega)^n X(\omega)$
Differentiation in frequency	$(-jt)^n x(t)$	$\frac{d^n}{d\omega^n}X(\omega)$
Integration in time	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega)$, assuming $X(0) = 0$.
Convolution in time	$(x * y)(t)$	$X(\omega)Y(\omega)$
Convolution in frequency	$x(t)y(t)$	$\frac{1}{2\pi}(X * Y)(\omega)$
Conjugate	$x^*(t)$	$X^*(-\omega)$
Conjugate, time-reversed	$x^*(-t)$	$X^*(\omega)$
Conjugate symmetry	$x(t)$ real-valued	$X(\omega) = X^*(-\omega)$ which implies $\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$
	$x(t)$ real and even i.e., $x(t) = x(-t)$	$X(\omega)$ real and even i.e., $X(\omega) = X(-\omega)$
Parseval's Equality	$\int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lvert X(\omega)\rvert^2 d\omega$	

Appendix 4.B Continuous-Time Fourier Transform : Pairs

	Signal	**Fourier transform**
Dirac delta function	$x(t) = \delta(t)$ $x(t) = \delta(t - t_0)$	$X(\omega) = 1$ $X(\omega) = e^{-jt_0\omega}$
Dirac comb	$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$X(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$
Constant function Harmonics	$x(t) = 1$ $x(t) = e^{j\omega_0 t}$ $x(t) = \cos(\omega_0 t)$ $x(t) = \sin(\omega_0 t)$	$X(\omega) = 2\pi\delta(\omega)$ $X(\omega) = 2\pi\delta(\omega - \omega_0)$ $X(\omega) = \pi\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$ $X(\omega) = \frac{\pi}{j}\left(\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right)$
Step function	$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0. \end{cases}$	$U(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$
One-sided exponential with $Re(a) > 0$ for integers $n \geq 2$	$x(t) = e^{-at}u(t)$ $x(t) = \frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$X(\omega) = \frac{1}{a + j\omega}$ $X(\omega) = \frac{1}{(a + j\omega)^n}$
Two-sided exponential	$x(t) = e^{-a\|t\|}$ with $Re(a) > 0$	$X(\omega) = \frac{2a}{a^2 + \omega^2}$
Sinc function	$x(t) = \sqrt{\frac{\omega_0}{2\pi}}\, \mathrm{sinc}\left(\frac{\omega_0}{2\pi}t\right)$ where $\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$	$X(\omega) = \begin{cases} \sqrt{\frac{2\pi}{\omega_0}}, & \|\omega\| \leq \frac{1}{2}\omega_0 \\ 0, & \text{otherwise.} \end{cases}$
Box function	$b(t) = \begin{cases} \frac{1}{\sqrt{t_0}}, & \|t\| \leq \frac{1}{2}t_0, \\ 0, & \text{otherwise.} \end{cases}$	$B(\omega) = \sqrt{t_0}\, \mathrm{sinc}\left(\frac{t_0}{2\pi}\omega\right)$

Appendix 4.C Discrete-Time Fourier Transform : Properties

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega \quad \longleftrightarrow \quad X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

Property	**Signal**	**Fourier transform**
Linearity	$\alpha x[n] + \beta y[n]$	$\alpha X(\omega) + \beta Y(\omega)$
Shift in time	$x[n - n_0]$	$e^{-j\omega n_0}X(\omega)$
Shift in frequency	$e^{j\omega_0 n}x[n]$	$X(\omega - \omega_0)$
Time Reversal	$x[-n]$	$X(-\omega)$
Differentiation in Frequency	$nx[n]$	$j\dfrac{dX(\omega)}{d\omega}$
Convolution in time	$(x * y)[n]$	$X(\omega)Y(\omega)$
Circular convolution in frequency	$x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{2\pi} X(\theta)Y(\omega - \theta))d\theta$
Conjugate	$x^*[n]$	$X^*(-\omega)$
Conjugate, time-reversed	$x^*[-n]$	$X^*(\omega)$
Conjugate symmetry	$x[n]$ real-valued	$X(\omega) = X^*(-\omega)$ which implies $\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$
	$x[n]$ real and even *i.e.*, $x[n] = x[-n]$	$X(\omega)$ real and even *i.e.*, $X(\omega) = X(-\omega)$

Parseval's Relation for Aperiodic Signals

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(\omega)|^2 d\omega$$

Appendix 4.D Discrete-Time Fourier Transform : Pairs

	Signal	**Fourier transform**
Kronecker delta	$\delta[n]$ $\delta[n-n_0]$	1 $e^{-jn_0\omega}$
Constant	$x[n]=1$	$2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega-2\pi l)$
Harmonics	$x[n]=e^{j\omega_0 n}$	$2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega-\omega_0-2\pi l)$
Step function	$u[n]=\begin{cases}1, & n\geq 0\\ 0, & \text{otherwise.}\end{cases}$	$\frac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{+\infty}\pi\delta(\omega-2\pi k)$
One-sided exponential	$x[n]=\begin{cases}\alpha^n, & n\geq 0\\ 0, & \text{otherwise.}\end{cases}$ with $\lvert\alpha\rvert<1$	$\frac{1}{1-\alpha e^{-j\omega}}$
"Arithmetic- -geometric"	$x[n]=\begin{cases}n\alpha^n, & n\geq 0\\ 0, & \text{otherwise.}\end{cases}$ with $\lvert\alpha\rvert<1$	$\frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$
Sinc sequence	$\sqrt{\frac{\omega_0}{2\pi}}\operatorname{sinc}\left(\frac{\omega_0}{2\pi}n\right)$ where $\operatorname{sinc}(x)=\frac{\sin\pi x}{\pi x}$	$X(\omega)=\begin{cases}\sqrt{\frac{2\pi}{\omega_0}}, & \lvert\omega\rvert\leq\frac{1}{2}\omega_0\\ 0, & \text{otherwise.}\end{cases}$
Box sequence	$x[n]=\begin{cases}\frac{1}{\sqrt{n_0}}, & \lvert n\rvert\leq\frac{1}{2}(n_0-1),\\ 0, & \text{otherwise.}\end{cases}$ where n_0 is odd	$\sqrt{n_0}\frac{\operatorname{sinc}\left(\frac{n_0}{2\pi}\omega\right)}{\operatorname{sinc}\left(\frac{1}{2\pi}\omega\right)}$

Appendix 4.E Continuous-time Periodic Signals

Although we will not have time to cover the material of this Appendix in class, we include it in the lecture notes since you have seen Fourier series in your Analysis classes.

4.E.1 The Fourier Series

Consider a signal $x(t)$ with fundamental period T that can be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j\frac{2\pi}{T}kt}, \tag{4.38}$$

for arbitrary complex coefficients X_k. The so-called *fundamental frequency* of this signal is $\omega_0 = \frac{2\pi}{T}$. As you learn in your Analysis III-IV courses, there is actually a small miracle happening here: *All* well-behaved T-periodic functions can be expressed in this fashion! (In your Analysis classes, you learn more about which signals qualify as "well-behaved.")

The coefficients X_k for $-\infty < k < \infty$ are referred to as the *Fourier series* of the periodic signal $x(t)$. For any T-periodic signal $x(t)$, these coefficients are simply given by

$$X_k = \frac{1}{T}\int_0^T x(t)e^{-j\frac{2\pi}{T}kt}dt. \tag{4.39}$$

You see a proof of this in your Analysis classes.

4.E.2 Properties of the Fourier Series

The Fourier series coefficients have many interesting properties, some almost trivial, others rather intricate. A table of these properties is given in Appendices 4.F and 4.G.

In these notes, let us only pull out a single property, usually referred to as the *Parseval* identity. As we saw, for a periodic signal, the power can be calculated easily as

$$\mathcal{P} = \frac{1}{T}\int_0^T |x(t)|^2 dt. \tag{4.40}$$

A question of obvious engineering interest is whether we can also calculate the power directly from the Fourier series X_k (i.e., without first retrieving the signal $x(t)$ via Equation (4.38) and then integrating the square). Indeed, the Parseval identity asserts that

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X_k|^2. \tag{4.41}$$

This has an intuitively pleasing intuition: $|X_k|^2$ characterizes the power of the signal $x(t)$ at frequency $k \cdot \frac{2\pi}{T}$. Parseval's relation then says that the overall power of the signal is simply the sum of the powers at the individual frequencies.

To prove this is surprisingly simple: We write out

$$\begin{aligned} \frac{1}{T}\int_0^T |x(t)|^2 dt &= \frac{1}{T}\int_0^T \left| \sum_{k=-\infty}^{\infty} X_k e^{j\frac{2\pi}{T}kt} \right|^2 dt \\ &= \frac{1}{T}\int_0^T \left(\sum_{k=-\infty}^{\infty} X_k e^{j\frac{2\pi}{T}kt} \right) \left(\sum_{m=-\infty}^{\infty} X_m e^{j\frac{2\pi}{T}mt} \right)^* dt \\ &= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} X_k X_m^* \frac{1}{T}\int_0^T e^{j\frac{2\pi}{T}kt} e^{-j\frac{2\pi}{T}mt} dt. \end{aligned} \tag{4.42}$$

The key now is to observe that

$$\frac{1}{T}\int_0^T e^{j\frac{2\pi}{T}(k-m)t} dt = \begin{cases} 1, & \text{if } k = m, \\ 0, & \text{otherwise,} \end{cases} \tag{4.43}$$

which you can verify easily by recalling that $e^{jx} = \cos(x) + j\sin(x)$. In words, this says that the functions $\frac{1}{\sqrt{T}} e^{j\frac{2\pi}{T}kt}$ are *orthonormal.* Using this in Equation (4.42) completes the proof.

4.E.3 LTI Systems and Fourier Series

It is immediately clear that if we pass a signal of the form of Equation (4.38) through an LTI system with frequency response $H(\omega)$, the output is

$$\begin{aligned} y(t) &= \mathcal{H}\left\{ \sum_{k=-\infty}^{\infty} X_k e^{j\frac{2\pi}{T}kt} \right\} \\ &= \sum_{k=-\infty}^{\infty} X_k \mathcal{H}\left\{ e^{j\frac{2\pi}{T}kt} \right\} \\ &= \sum_{k=-\infty}^{\infty} X_k H\left(j\frac{2\pi}{T}k \right) e^{j\frac{2\pi}{T}kt}, \end{aligned} \tag{4.44}$$

where the second equality follows by linearity, and the last equality is Equation (5.2). By analogy to Equation (4.38), the Fourier series representation of the signal $y(t)$ is given by

$$y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{j\frac{2\pi}{T}kt}. \tag{4.45}$$

Comparing this with Equation (4.44), we see that

$$Y_k = H\left(\frac{2\pi}{T}k \right) X_k. \tag{4.46}$$

That is, when we are studying periodic input signals and LTI systems, it is advantageous to represent the input signal by its Fourier series and the LTI system by its frequency response. In this way, rather than having to solve a convolution integral, we can perform the system operation in the frequency domain according to Equation (4.46) by a simple point-wise multiplication, separately for each frequency.

Appendix 4.F Continuous-Time Fourier Series : Properties

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} X_k e^{jk(2\pi/T)t}$$

$$X_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

Let $x(t)$ and $y(t)$ both be periodic signals of fundamental period T.

Property	**Periodic Signal**	**Fourier Series Coefficients**
Linearity	$Ax(t) + By(t)$	$AX_k + BY_k$
Time-Shifting	$x(t - t_0)$	$X_k e^{-jk\omega_0 t_0} = X_k e^{-jk(2\pi/T)t_0}$
Frequency-Shifting	$e^{jM(2\pi/T)t}x(t)$	X_{k-M}
Conjugation	$x^*(t)$	X^*_{-k}
Time Reversal	$x(-t)$	X_{-k}
Time Scaling	$x(\alpha t), \alpha > 0$	X_k
Periodic Convolution	$\int_T x(\tau)y(t-\tau)d\tau$	TX_kY_k
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} X_l Y_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 X_k = jk\frac{2\pi}{T}X_k$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\left(\frac{1}{jk\omega_0}\right)X_k = \left(\frac{1}{jk(2\pi/T)}\right)X_k$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X_k = X^*_{-k} \\ Re(X_k) = Re(X_{-k}) \\ Im(X_k) = -Im(X_{-k}) \\ \lvert X_k\rvert = \lvert X_{-k}\rvert \\ \arg X_k = -\arg X_{-k} \end{cases}$
Real and Even Signals	$x(t)$ real and even	X_k real and even
Real and Odd Signals	$x(t)$ real and odd	X_k purely imaginary and odd

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |X_k|^2$$

Appendix 4.G Continuous-Time Fourier Series : Pairs

Signal	**Fourier series coefficients**
$\sum_{k=-\infty}^{+\infty} X_k e^{jk\omega_0 t}$	X_k
$e^{j\omega_0 t}$	$X_1 = 1$ $X_k = 0, \quad \text{otherwise}$
$\cos \omega_0 t$	$X_1 = X_{-1} = \frac{1}{2}$ $X_k = 0, \quad \text{otherwise}$
$\sin \omega_0 t$	$X_1 = -X_{-1} = \frac{1}{2j}$ $X_k = 0, \quad \text{otherwise}$
$x(t) = 1$	$X_0 = 1, \quad X_k = 0,\ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\frac{\omega_0 T_1}{\pi} \text{ sinc } \left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$X_k = \frac{1}{T}$ for all k

Chapter 5

The Frequency Response of Stable LTI Systems

As we saw in previous chapter, the frequency-domain characterization of an LTI system in terms of its frequency response is an alternative to the time-domain characterization through convolution and impulse response. It is particularly convenient to analyze LTI systems in the frequency-domain because differential equations, difference equations, and convolution operations in the time domain all become algebraic operations in the frequency domain. In this chapter, we will continue to develop the frequency-domain perspective by studying frequency-selective filtering and sampling. The culmination of the tools developed so far will be the *sampling theorem*, a surprising and elegant result which states that under certain conditions a continuous-time signal could be represented with discrete-time samples without any loss of information!

5.1 Continuous-time Systems

5.1.1 The Frequency Response

Let us suppose that the input to our stable LTI system is given by

$$x(t) = e^{j\omega_0 t}, \tag{5.1}$$

which is also called a *complex exponential of (angular) frequency* ω_0. Note that it is a periodic signal of fundamental period $T = \frac{2\pi}{\omega_0}$. Then, the output is given by

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} e^{j\omega_0(t-\tau)} h(\tau) d\tau = e^{j\omega_0 t} \underbrace{\int_{-\infty}^{\infty} e^{-j\omega_0 \tau} h(\tau) d\tau}_{H(\omega_0)} \\ &= H(\omega_0) e^{j\omega_0 t}, \end{aligned} \tag{5.2}$$

as long as the integral converges. Now, if we assume that the LTI system is stable, which means that $\int_{-\infty}^{\infty} |h(t)| dt$ converges, then it is easy to show that the integral in the definition of $H(\omega_0)$ converges, too. Recall, we call $H(\omega_0)$ the *frequency response* of our LTI system at frequency ω_0.

It is important to pause at this point and fully digest this result: For *any* LTI system, if the input is a complex exponential of frequency ω_0, then the output is again a complex exponential of exactly the same frequency ω_0. The only thing that the LTI system does to the complex exponential is to change its amplitude (namely, by $|H(\omega_0)|$) and its phase (namely, by $\arg(H(\omega_0))$).

One powerful interpretation of this is that complex exponentials are *eigenfunctions* of all LTI systems. To understand this, recall from your class on linear algebra that the matrix-vector product $A\mathbf{x}$ is, in general, cumbersome to evaluate — except if $\mathbf{x}$ is an *eigenvector* of the matrix A. In this case, we have that $A\mathbf{x} = \lambda\mathbf{x}$, where λ is the eigenvalue corresponding to the eigenvector $\mathbf{x}$. To return to LTI systems, the eigenvalue corresponding to the "eigenvector" $e^{j\omega_0 t}$ is simply the frequency response evaluated at ω_0, namely, $H(\omega_0)$, as a comparison with Equation (5.2) reveals.

5.1.2 Interpretation of the Frequency Response

The frequency response is a powerful system representation in part because it admits very natural insights into the behavior of the system.

The first interpretation concerns the *magnitude* $|H(\omega)|$ of the frequency response. A simple plot of this magnitude against the frequency ω already gives a good idea of what kind of a system we are dealing with. For example, some systems will have a large value of $|H(\omega)|$ for small frequencies ω, but a much smaller value (perhaps even zero) for large frequencies. Such a system is referred to as a *low-pass filter,* because it only passes the low frequencies, but significantly weakens (or even fully blocks) the high frequencies. In a similar fashion, there are *high-pass filters, band-pass filters* (these pass only a certain intermediate range of frequencies, but block both low and high frequencies), *band-stop (or notch) filters* (these block a certain intermediate range of frequencies, while passing both low and high frequencies), and so on. It should be noted that this filter terminology is not of the rigorous mathematical kind; rather, it is approximate engineering parlance (but it turns out to be very useful).

The second interpretation concerns the *phase* $\arg H(\omega)$. Again, if the input signal is the complex exponential $x(t) = e^{j\omega t}$, the output is

$$\begin{aligned} y(t) &= H(\omega)e^{j\omega t} \\ &= |H(\omega)|e^{j\arg H(\omega)}\ e^{j\omega t} \\ &= |H(\omega)|e^{j\omega(t+\arg H(\omega)/\omega)}. \end{aligned} \tag{5.3}$$

That is, the phase can be thought of as a *delay* by $-\arg H(\omega)/\omega$. It should be noted that each frequency will be delayed by a different time shift (except in the practically interesting case of so-called *linear phase,* which means that $\arg H(\omega) = -t_0\omega$, where the delay is exactly t_0 irrespective of the frequency).

5.1.3 Basic Properties of the Frequency Response

1. We consider both positive and negative values of ω_0 since clearly, the main insight we obtained in Equation (5.2) holds for both cases.

   At the same time, it might be a little confusing that we are considering "negative" frequencies. To clarify things, let us consider the frequency $-\omega_0$ (where ω_0 itself is a positive number). That is, we consider the signal

   $$x(t) \quad = \quad e^{-j\omega_0 t}. \tag{5.4}$$

   If the goal is to gain insight and interpretability, it is often a good idea to express complex numbers in terms of their real and imaginary parts, which gives:

   $$x(t) \quad = \quad \cos(-\omega_0 t) + j\sin(-\omega_0 t). \tag{5.5}$$

   But remembering from calculus that $\cos(-\omega_0 t) = \cos(\omega_0 t)$ and $\sin(-\omega_0 t) = -\sin(\omega_0 t)$, this is exactly the same as

   $$x(t) \quad = \quad \cos(\omega_0 t) - j\sin(\omega_0 t), \tag{5.6}$$

   and this expression only contains positive frequencies. Thus, negative frequencies do not need to be interpreted in a physical way, they are only a tool to make the mathematical expressions more compact.

2. When the impulse response $h(t)$ of the system is *real-valued,* the frequency response satisfies

   $$H(\omega_0) \quad = \quad H^*(-\omega_0), \tag{5.7}$$

   for every ω_0, where * denotes the complex conjugate. One often says that in this case, the frequency response is *conjugate-symmetric.*

5.1.4 Why do we consider complex-valued signals?

Why did we use a *complex exponential* as the test input? — We do not really have to do this. Instead, we could have considered the real-valued test input $x(t) = \cos(\omega_0 t)$. Here, too, something nice happens. But as we will see, formulas get a little more cumbersome, which is why the complex exponentials in the end are more convenient to deal with.

More precisely, if the system input is $x(t) = \cos(\omega_0 t)$, the output is equally simple to figure out. To see this, it is enough to recall the Euler formula: $x(t) = \cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$. Since the system is linear, the output is just the sum of the outputs corresponding to the two summands, hence $y(t) = \frac{1}{2}H(\omega_0)e^{j\omega_0 t} + \frac{1}{2}H(-\omega_0)e^{-j\omega_0 t}$.

How to interpret this formula? First of all, since we are considering a real-valued input, we probably also want to assume that the system impulse response $h(t)$ is real-valued. But then, the system output $y(t)$ must be real-valued, simply from Formula (3.10). At a first glance, the expression we found for the output signal $y(t)$ does not appear to be real-valued. How can we understand that in spite of appearance, it actually *is* real-valued?

To see this, we exploit the property that if $h(t)$ is real-valued, we must have that $H(\omega_0) = H^*(-\omega_0)$. Using polar coordinates, we can write $H(\omega_0) = |H(\omega_0)|e^{j \arg H(\omega_0)}$ and thus, $H^*(\omega_0) = |H(\omega_0)|e^{-j \arg H(\omega_0)}$. But then, we observe

$$\begin{aligned} y(t) &= \frac{1}{2}H(\omega_0)e^{j\omega_0 t} + \frac{1}{2}H(-\omega_0)e^{-j\omega_0 t} \\ &= \frac{1}{2}H(\omega_0)e^{j\omega_0 t} + \frac{1}{2}H^*(\omega_0)e^{-j\omega_0 t} \\ &= |H(\omega_0)|\frac{e^{j \arg H(\omega_0)}e^{j\omega_0 t} + e^{-j \arg H(\omega_0)}e^{-j\omega_0 t}}{2} \\ &= |H(\omega_0)| \cos(\omega_0 t + \arg H(\omega_0)), \end{aligned} \tag{5.8}$$

where, for the last step, we have again used the Euler formula. What this says is that if the system input is $x(t) = \cos(\omega_0 t)$, then the output is again a cosine of exactly the same frequency. However, it no longer has amplitude 1, but rather, the amplitude is given by $|H(\omega_0)|$. Moreover, it is also phase-shifted by $\arg H(\omega_0)$ (meaning that there is no longer a peak at $t = 0$, but there now is a peak at $t = -\frac{\arg H(\omega_0)}{\omega_0}$).

An alternative derivation is found by considering cartesian (rather than polar) coordinates, which is left as an exercise for the reader.

In summary, by comparing Equation (5.8) to the much simpler expression in Equation (5.2), it should be clear that dealing with complex exponentials is much more convenient than dealing with cosines.

5.1.5 Composition of Systems

As we have seen earlier, there are two natural ways of composing two systems: parallel and series. In both cases, it is a simple matter to find the frequency response of the composed system from the frequency responses of the individual systems.

In particular, let us consider two systems with frequency responses $H_1(\omega)$ and $H_2(\omega)$, respectively, composed in parallel exactly as in Figure 3.2. Then, the overall system has frequency response

$$G(\omega) = H_1(\omega) + H_2(\omega) \tag{5.9}$$

since integration is linear. If instead these systems are composed in series like in Figure 3.3, then we can use Equation (3.33) to infer that

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} e^{-j\omega_0 t}\left(\int_{-\infty}^{\infty} h_1(\tau)h_2(t-\tau)d\tau\right)dt \\ &= H_1(\omega)H_2(\omega), \end{aligned} \tag{5.10}$$

which follows from the usual tricks (inverting integration order and change of variable inside the integral). Recall that this is also known as the "convolution property" of the Fourier transform .

5.2 Discrete-time Systems

5.2.1 The Frequency Response

Let us suppose that the input to our stable LTI system is given by

$$x[n] \quad = \quad e^{j\omega_0 n}, \tag{5.11}$$

which is called a *complex exponential of (angular) frequency* ω_0. Then, the output is given by

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} e^{j\omega_0(n-k)} h[k] = e^{j\omega_0 n} \underbrace{\sum_{k=-\infty}^{\infty} e^{-j\omega_0 k} h[k]}_{H(\omega_0)} \\ &= H(\omega_0) e^{j\omega_0 n}, \end{aligned} \tag{5.12}$$

as long as the (doubly) infinite sum converges. Now, if we assume that the LTI system is stable, which means that $\sum_{k=-\infty}^{\infty} |h[k]|$ converges, then it is easy to show that the sum in the definition of $H(\omega_0)$ converges, too. Recall that we call $H(\omega_0)$ the *frequency response* of our LTI system at frequency ω_0.

Just like in the continuous-time case, this powerful formula says that if the input to any LTI system is a complex exponential of frequency ω_0, then so is the output. The only change introduced by the LTI system concerns the amplitude and the phase of the complex exponential.

Hence, in the same fashion as in the continuous-time case, one can think of the complex exponentials $e^{j\omega_0 n}$ as eigenfunctions of all (discrete-time) LTI systems.

We should also add a short note concerning the somewhat unusual notation $H(\omega_0)$. This notation conveniently stresses that the discrete-time Fourier transform is a 2π-periodic function. Indeed, you can easily verify in Equation (5.12) that if instead of ω_0, you consider the frequency $\omega_0 + \ell 2\pi$ for any integer ℓ, you obtain exactly the same answer: $H(\omega_0) = H(\omega_0 + \ell 2\pi))$.

5.2.2 Interpretation of the Frequency Response

The interpretation of the frequency response proceeds along the same lines as in the continuous-time case, but keeping in mind that only the frequency interval $(-\pi, \pi]$ needs to be considered (since the frequency response is 2π-periodic).

5.2.3 Basic Properties of the Frequency Response

1. The frequency response is 2π-periodic.

   This fact is a simple consequence of the frequency response being the discrete-time Fourier transform of the impulse response.

2. When the impulse response $h[n]$ of the system is *real-valued,* the frequency response satisfies

$$H(\omega_0) \quad = \quad H^*(-\omega_0), \tag{5.13}$$

for every ω_0, where * denotes the complex conjugate. One often says that in this case, the frequency response is *conjugate-symmetric.*

3. Following up on this insight, we can again show that if the system's impulse response $h[n]$ is real-valued and if the input to the system is $x[n] = \cos(\omega_0 n)$, then the corresponding output is given by

$$y[n] = |H(\omega_0)| \cos(\omega_0 n + \arg(H(\omega_0))), \tag{5.14}$$

using the same exact steps as in Subsection 5.1.4.

5.2.4 Composition of Systems

As we have seen earlier, there are two natural ways of composing two systems: parallel and series. In both cases, it is a simple matter to find the frequency response of the composed system from the frequency responses of the individual systems.

In particular, let us consider two systems with frequency responses $H_1(\omega)$ and $H_2(\omega)$, respectively, composed in parallel exactly as in Figure 3.2. The overall frequency response is given by

$$G(\omega) \quad = \quad H_1(\omega) + H_2(\omega) \tag{5.15}$$

since the overall sum can be broken into two separate sums. If instead these systems are composed in series like in Figure 3.3, then we can use Equation (3.33) to infer that

$$\begin{aligned} G(\omega) \quad &= \quad \sum_{n=-\infty}^{\infty} e^{-j\omega_0 n} \sum_{k=-\infty}^{\infty} h_1[k] h_2[n-k] && (5.16) \\ &= \quad H_1(\omega) H_2(\omega), && (5.17) \end{aligned}$$

which follows from the usual tricks (inverting summation order and change of variable inside the integral). Again, this is the "convolution property" of the discrete-time Fourier transform.

5.3 Sampling

We will use the tools developed so far to analyze how to represent a continuous-time signal with equally spaced discrete-time *samples.* A somewhat surprising result known as the *sampling theorem* states that under certain conditions a continuous-time signal can be completely recovered from a sequence of its samples. The sampling theorem is extremely important and useful since processing discrete-time signals is more flexible and is often preferable to processing continuous-time signals.

5.3.1 Impulse-Train Sampling

One way in which the sampling operation may be represented mathematically is with *impulse-train sampling.* This is done by multiplying the continuous-time signal $x(t)$ that we wish to sample by a periodic impulse train. In the time domain,

$$x_p(t) = x(t)p(t) \tag{5.18}$$

where

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT). \tag{5.19}$$

The periodic impulse train $p(t)$ is referred to as the *sampling function*, the period T is the *sampling period*, and fundamental frequency of $p(t)$, $\omega_s = \frac{2\pi}{T}$, as the *sampling frequency.*

Combining (5.18) with (5.19) we see that $x_p(t)$ is an impulse train with the amplitudes of the impulses equal to the samples of $x(t)$ at intervals spaced by T. That is,

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT). \tag{5.20}$$

We emphasize that (5.18) represents a continuous-time system with an input $x(t)$ and output $x_p(t)$. A discrete-time signal of samples could also be constructed by letting $x[n] = x(nT)$. The values of the discrete-time signal $x[n]$ are the same as the amplitudes of the impulses of $x_p(t)$ at nT. We can think of $x_p(t)$ as an equivalent representations of a sampled signal $x[n]$; in other words, $x_p(t)$ and $x[n]$ contain the same amount of information about the original signal $x(t)$.

To understand the impact of sampling on the continuous-time signal $x(t)$ we will keep analyzing the continuous-time system (5.18) in the frequency domain. From the *convolution in frequency* property, we know that

$$X_p(\omega) = \frac{1}{2\pi}(X * P)(\omega). \tag{5.21}$$

From Appendix 4.B using the *Dirac comb* Fourier pair,

$$P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s). \tag{5.22}$$

Combining (5.21) with (5.22) and using the distributive property of convolution we compute,

$$X_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s). \tag{5.23}$$

That is, $X_p(\omega)$ is a periodic function of ω consisting of a superposition of shifted replicas of $X(\omega)$, scaled by $\frac{1}{T}$. Suppose that there is a frequency, ω_M, such that $X(\omega) = 0$, whenever

$|\omega| > \omega_M$. If $\omega_M < (\omega_s - \omega_M)$, or equivalently, $\omega_s > 2\omega_M$, then there is no overlap between the shifted replicas of $X(\omega)$, whereas with $\omega_s < 2\omega_M$, there is overlap. Consequently, if $\omega_s > 2\omega_M$, $x(t)$ can be recovered exactly from $x_p(t)$ using a lowpass filter with gain T and a cutoff frequency greater than ω_M and less than $\omega_s - \omega_M$.

To fully understand this, it is key to draw a picture — which the reader is encouraged to do, and which we will do in lecture.

5.3.2 The Sampling Theorem

By modeling the sampling operation with impulse-train sampling we arrived at the sampling theorem. Succinctly, the sampling theorem can be stated as follows.

Let $x(t)$ be a band-limited signal with $X(\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm 1, \pm 2, \ldots$, if

$$\omega_s > 2\omega_M, \tag{5.24}$$

where

$$\omega_s = \frac{2\pi}{T}. \tag{5.25}$$

Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter with gain T and cutoff frequency greater than ω_M and less than $\omega_s - \omega_M$. The resulting output signal will exactly equal $x(t)$.

The frequency $2\omega_M$, which, under the sampling theorem, must be exceeded by the sampling frequency, is commonly referred to as the *Nyquist rate.* (The frequency ω_M corresponding to one-half the Nyquist rate is often referred to as the *Nyquist frequency.*)

5.3.3 Sampling a Complex Exponential

In previous sections we derived the sampling theorem by representing the sampling operation as a multiplication by an impulse train. In this section, we consider what happens as we sample a continuous-time complex exponential. Before we do this, it is important to recall the following helpful fact about discrete-time complex exponentials: a discrete-time exponential is 2π-periodic as a function of ω. That is

$$e^{j\omega n} = e^{j\omega n} e^{j2\pi n} = e^{j(\omega + 2\pi)n} \tag{5.26}$$

holds for every integer n. More generally, this holds for any multiple of 2π. That is,

$$e^{j\omega n} = e^{j\omega n} e^{j2\pi l n} = e^{j(\omega + 2\pi l)n} \tag{5.27}$$

for $l = 0, \pm 1, \pm 2, \ldots$.

Suppose now that we sample a continuous-time signal

$$x(t) = e^{j\omega_0 t} \tag{5.28}$$

with sampling frequency ω_s. The resulting discrete-time signal is

$$x[n] \quad = \quad e^{j\omega_0 nT}, \tag{5.29}$$

where $T = \frac{2\pi}{\omega_s}$. By defining $\tilde{\omega} = 2\pi\frac{\omega_0}{\omega_s}$ we can rewrite $x[n]$ as

$$x[n] \quad = \quad e^{j2\pi\frac{\omega_0}{\omega_s}n} = e^{j\tilde{\omega}n}. \tag{5.30}$$

In light of our discussion above about periodicity of discrete-time complex exponentials we can also write

$$x[n] \quad = \quad e^{j(\tilde{\omega}+2\pi l)n}, \text{ for } l = 0, \pm 1, \pm 2, \ldots \tag{5.31}$$

But, the fact that we can rewrite the same signal $x[n]$ in all these different ways has an important implication to our discussion about sampling. It implies that if we sample any continuous-time signal of the form

$$\tilde{x}(t) \quad = \quad e^{j(\omega_0+\omega_s l)t} \tag{5.32}$$

with sampling frequency ω_s the resulting discrete-time signal will always be the same $x[n]$. At this point, we may wonder which continuous-time complex exponential is the correct reconstruction given samples $x[n]$? A reasonable answer is to pick the one with the smallest magnitude frequency. That is, we could find the value of l that minimizes the quantity $|\omega_0 + \omega_s l|$. For the impulse-train sampling discussed in the previous section, this is exactly what a low-pass filter reconstruction procedure does.

Going back to (5.28), we see that if $|\omega_s| > 2|\omega_0|$, then $|\tilde{\omega}| < \pi$ and picking the signal with lowest magnitude frequency would recover $x(t)$. If $|\omega_s| < 2|\omega_0|$, then $|\tilde{\omega}| > \pi$, and the reconstruction procedure would produce a different complex exponential.

5.3.4 Signal Reconstruction and Aliasing

As we have seen, for a band-limited signal, if the sampling instants are sufficiently close, then the signal can be reconstructed exactly by using an ideal low-pass filter. In the time domain, the impulse response of this lowpass filter is

$$h(t) = \frac{\omega_c T \sin \omega_c t}{\pi \omega_c t} \tag{5.33}$$

where T is the gain of the filter and ω_c is the cutoff frequency. Letting $x_r(t)$ denote the reconstructed signal we obtain

$$x_r(t) \quad = \quad (x_p * h)(t) \tag{5.34}$$

or

$$x_r(t) \quad = \quad \sum_{n=-\infty}^{\infty} x(nT)h(t-nT). \tag{5.35}$$

The reconstruction procedure in the time domain becomes

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\frac{\omega_c T}{\pi}\frac{\sin \omega_c(t-nT)}{\omega_c(t-nT)}. \tag{5.36}$$

The reconstruction using the impulse response of an ideal lowpass filter is commonly referred to as *band-limited interpolation*, since it implements exact reconstruction if $x(t)$ is band limited and the sampling frequency satisfies the conditions of the sampling theorem.

So far we assumed that the sampling frequency was sufficiently high that the conditions of the sampling theorem were met. What happens if this is not the case? The spectrum of the sampled signal consists of scaled replication of the spectrum of $x(t)$, and this forms the basis of the sampling theorem. When $\omega_s < 2\omega_M$, the spectrum of $x(t)$ overlaps in $X_p(\omega)$ and thus is no longer recoverable by lowpass filtering. This effect, in which the individual terms overlap, is referred to as *aliasing*.

Chapter 6

The Transfer Function and The Laplace Transform

As we saw, every LTI system can be characterized by its impulse response. If the system is stable, then there is a very insightful characterization in terms of the frequency response. This admits many intuitive interpretations.

What if the system is not stable? As we will now see, we can still apply a similar trick as before. In this way, we obtain a "generalized" frequency response, often referred to as the *transfer function* or *system function.* The latter does not quite have the same physically intuitive significance as the frequency response, but it will turn out to be an extremely versatile tool to understand general LTI systems.

The tool introduced in this section is the Laplace transform, which bears many similarities to the Fourier transform but uses a general complex number s rather than frequency.

6.1 The Transfer Function

Let us suppose that the input to our LTI system is given by

$$x(t) = e^{st}, \tag{6.1}$$

for an arbitrary *complex-valued* constant s. Then, the output is given by

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} e^{s(t-\tau)} h(\tau) d\tau = e^{st} \underbrace{\int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau}_{H(s)} \\ &= H(s) e^{st}, \end{aligned} \tag{6.2}$$

as long as the integral converges.

The function $H(s)$ is called the *transfer function* (or sometimes the *system function*) of the considered LTI system. The fundamental insight in Equation (6.2) is of a very similar nature to the consideration in the case of the frequency response: The general exponential signal e^{st} is essentially left *unchanged* by the LTI system! The only change occurs in the magnitude and phase.

A second important observation is that if instead of an arbitrary complex number s, we consider a purely imaginary number $s = j\omega$, then we obtain exactly the frequency response. In this sense, the frequency response is a special case of the transfer function, obtained as $H(\omega) = H(s = j\omega)$. For this reason, we use exactly the same symbol $H(\cdot)$ (that is, the capital letter H) both for the frequency response and for the transfer function.

The convergence discussion relating to Equation (6.2) will be a major part of the next section. As we recall, in the case of the frequency response, the convergence only depended on the impulse response $h(t)$. By contrast, here, convergence depends both on the impulse response $h(t)$ *and* on the value of s. It is precisely this dependence that makes the Laplace transform a very useful tool to understand the stability of LTI systems.

6.2 The Laplace Transform

For a time-domain signal $x(t)$, the Laplace transform is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt, \qquad (6.3)$$

for those values of s for which this integral converges. That is, the transfer function $H(s)$ of a system is exactly the Laplace transform of its impulse response $h(t)$.

We observe that by only considering s of the form $s = j\omega$, that is, by evaluating the Laplace transform only along the *imaginary axis* in the complex s-plane, we obtain exactly the Fourier transform. In this sense, the Laplace transform is a strict generalization of the Fourier transform.

The convergence discussion relating to Equation (6.3) will be the subject of the next few subsections. The main insight is that for many cases of interest, the integral in Equation (6.3) will converge for certain values of s and will diverge for other values of s. As we will see, it is very instructive to *sketch* the region of those values of s for which the integral converges into the full complex s-plane. This is called the *Region of Convergence* (ROC).

To invert the Laplace transform, i.e., given $X(s)$ (and its corresponding region of convergence), find back the time-domain signal $x(t)$, we can formally write

$$x(t) = \int X(s)e^{st}ds, \qquad (6.4)$$

but this involves complex integration (over the variable s). Due to the time constraints imposed by this class, we will not get to cover this approach. Instead, we will invert the Laplace transform using the tables provided at the end of this chapter, in Appendices 6.A and 6.B

6.2.1 Key Example 1: Single Real-Valued Pole

To begin to understand and appreciate the Laplace transform, it is instructive to juxtapose the following two examples:

$$h_c(t) = \begin{cases} e^{-at}, & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{6.5}$$

where a is a real number. In the engineering literature, such a signal is sometimes called *right-sided* because the graph of the signal lives on the "right side" in the Euclidean plane. It is also sometimes called a *causal* signal since if you interpret this signal as the impulse response of an LTI system, then this system would be causal.

Moreover, this signal is well-behaved (more precisely, absolutely integrable) if $a > 0$. In the engineering literature, such a signal is also sometimes referred to as a *stable* signal since if we interpret the signal as the impulse response of an LTI system, then this system would be stable.

By contrast, if $a \leq 0$, the signal is not absolutely integrable, and we could refer to this as an *unstable* signal.

We find

$$\begin{aligned} H_c(s) &= \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau \\ &= \int_{0}^{\infty} e^{-(s+a)\tau} d\tau \\ &= \frac{1}{s+a}, \end{aligned} \tag{6.6}$$

as long as $Re(s) + a > 0$, i.e., as long as $Re(s) > -a$. Otherwise, the exponential inside the integral leading to Equation (6.2) diverges, and hence, the integral is not finite.

$$h_a(t) = \begin{cases} 0 & t > 0, \\ -e^{-at}, & t \leq 0, \end{cases} \tag{6.7}$$

where a is a real number. In the engineering literature, such a signal is sometimes called *left-sided* because the graph of the signal lives on the "left side" in the Euclidean plane. It is also sometimes called a *anti-causal* signal since if you interpret this signal as the impulse response of an LTI system, then this system would be anti-causal.

Moreover, this signal is well-behaved (more precisely, absolutely integrable) if $a < 0$. In the engineering literature, such a signal is also sometimes referred to as a *stable* signal since if we interpret the signal as the impulse response of an LTI system, then this system would be stable.

By contrast, if $a \geq 0$, the signal is not absolutely integrable, and we could refer to this as an *unstable* signal.

We find

$$\begin{aligned} H_a(s) &= \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau \\ &= -\int_{-\infty}^{0} e^{-(s+a)\tau} d\tau \\ &= \frac{1}{s+a}, \end{aligned} \tag{6.8}$$

as long as $Re(s) + a < 0$, i.e., as long as $Re(s) < -a$. Otherwise, the exponential inside the integral leading to Equation (6.2) diverges, and hence, the integral is not finite.

We observe that in both cases, the Laplace transform evaluates to the formula $H(s) = \frac{1}{s+a}$. This is often referred to as a *pole at* $-a$.

At this point, we should have doubts about the usefulness of the Laplace transform: Apparently, one and the same Laplace transform, in our example $H(s) = \frac{1}{s+a}$, can represent two very different time domain signals! As it stands, the Laplace transform cannot be inverted.

In order to have a meaningful transform, we have to add more information: We have to specify the range of values of s for which the formula is valid. This is usually referred to as the *region of convergence* (ROC).

For the example at hand, that is $H(s) = \frac{1}{s+a}$, it turns out that there are only two interesting choices of the region of covergence:

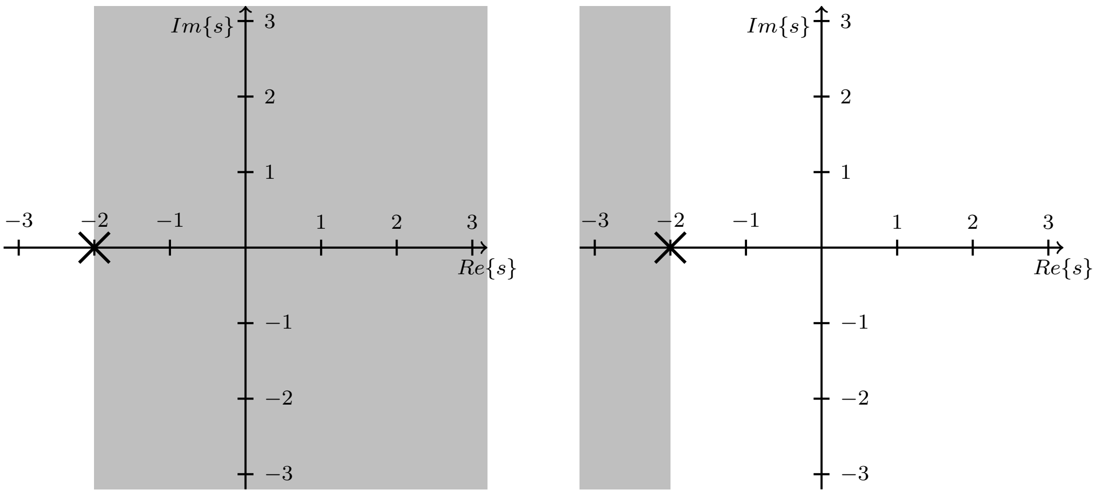
$$\begin{array}{ccc} \text{ROC}(H(s)) = \{s : Re(s) > -a\} & \text{or} & \text{ROC}(H(s)) = \{s : Re(s) < -a\} \\ \Downarrow & & \Downarrow \\ h(t) = \begin{cases} e^{-at}, & t \geq 0, \\ 0, & t < 0, \end{cases} & & h(t) = \begin{cases} 0, & t > 0, \\ -e^{-at}, & t \leq 0. \end{cases} \end{array} \tag{6.9}$$

By far the best way to use the concept of ROC is graphical. For the example at hand, this is illustrated in Figure 6.1 for $a = 2$ and in Figure 6.2 for $a = -2$.

Let us summarize the observations from these example figures concerning the ROC:

- The signal is "causal" (that is, right-sided) if the ROC extends indefinitely to the right.
- The signal is "anti-causal" (that is, left-sided) if the ROC extends indefinitely to the left.
- The time-domain signal is "stable" (that is, absolutely integrable) *if and only if* the ROC includes the *imaginary axis.* This is pleasing: As we observed, the Fourier transform is simply the Laplace transform, evaluated along $s = j\omega$, that is, along the imaginary axis. Hence, whenever the ROC includes the imaginary axis, this implies that the Fourier transform of the time-domain signal also exists. But as we have seen, the Fourier transform only exists for integrable time-domain signals.

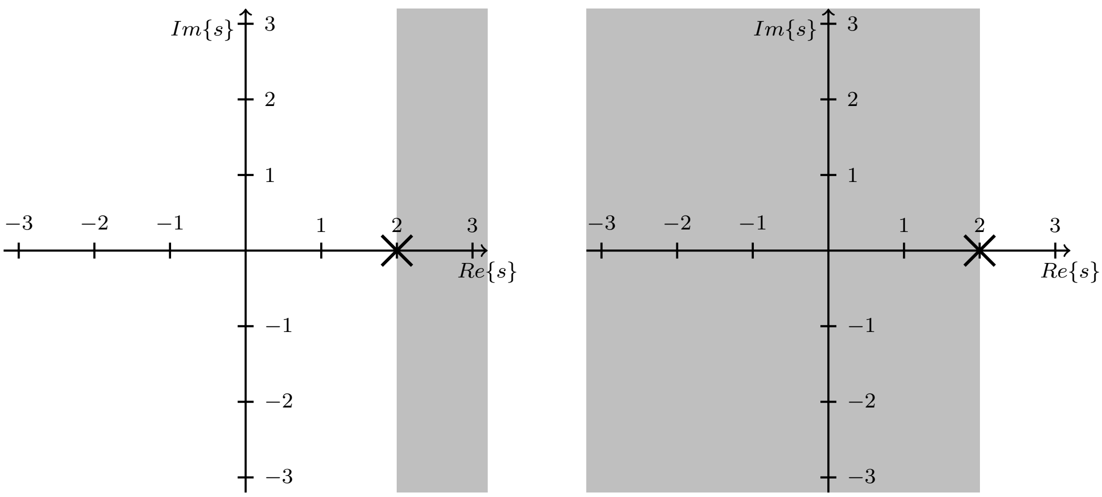
As we will show, these are fundamental properties of the ROC in full generality, not just for the examples considered here.



(a) Hence, $h(t) = \begin{cases} e^{-2t}, & t \geq 0, \\ 0, & t < 0, \end{cases}$
a "stable and causal" signal

(b) Hence, $h(t) = \begin{cases} 0, & t > 0, \\ -e^{-2t}, & t \leq 0, \end{cases}$
an "unstable and anti-causal" signal

Figure 6.1: The two regions of convergence for $H(s) = \frac{1}{s+2}$ (that is, $a = 2$).



(a) Hence, $h(t) = \begin{cases} e^{2t}, & t \geq 0, \\ 0, & t < 0, \end{cases}$
an "unstable and causal" signal

(b) Hence, $h(t) = \begin{cases} 0, & t > 0, \\ -e^{2t}, & t \leq 0, \end{cases}$
a "stable and anti-causal" signal

Figure 6.2: The two regions of convergence for $H(s) = \frac{1}{s-2}$ (that is, $a = -2$).

6.2.2 Key Example 2: Complex-Valued Pole

So far, we have considered a real-valued exponential. Let us now consider a complex-valued version, which we can write for example as

$$g(t) = \begin{cases} e^{-(a+j\omega_0)t}, & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{6.10}$$

where a and ω_0 are real numbers. This signal is complex-valued, which is not a problem, but here we prefer to consider the slightly more general case

$$h_c(t) = \begin{cases} e^{-(a+j\omega_0)t} + e^{-(a-j\omega_0)t} = 2e^{-at}\cos(\omega_0 t), & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{6.11}$$

which, as you can easily convince yourself, is again real-valued. Let us again observe that this is a well-behaved signal if $a > 0$, but it is not integrable if $a \leq 0$ (and, in fact, blows up exponentially fast if $a < 0$). For this signal, we find the Laplace transform

$$H_c(s) = \frac{1}{s + (a + j\omega_0)} + \frac{1}{s + (a - j\omega_0)} = \frac{2(s+a)}{(s + (a + j\omega_0))(s + (a - j\omega_0))}. \tag{6.12}$$

Hence, this Laplace transform has two (complex-conjugate) poles as illustrated in Figure 6.3 for the example $a = 2, \omega = 1$, and in Figure 6.4 for the example $a = -2, \omega = 1$. Moreover, this transfer function also has a *zero* at $s = -a$, that is, we have $H(s = -a) = 0$. This is often illustrated by a circle in the complex s-plane, leading to the so-called *pole-zero plot.*

But consider now the signal

$$h_a(t) = \begin{cases} 0, & t > 0, \\ -e^{-(a+j\omega_0)t} - e^{-(a-j\omega_0)t} = -2e^{-at}\cos(\omega_0 t), & t \leq 0. \end{cases} \tag{6.13}$$

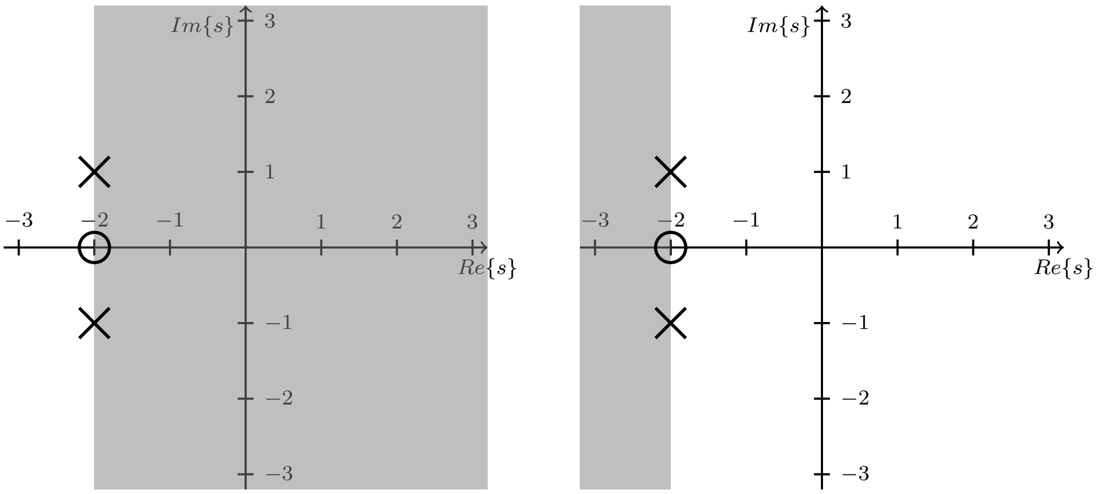
By analogy to the real-valued pole example, one can easily calculate the corresponding Laplace transform as

$$H_a(s) = \frac{2(s+a)}{(s + (a + j\omega_0))(s + (a - j\omega_0))}, \tag{6.14}$$

exactly the same as before.

So again, this Laplace transform represents two very different signals. In order to decide which of the two signals is meant, we need additional information in the shape of the ROC. Interestingly, the ROC only depends on the value of a. It does not hinge on the value of ω_0. More specifically, there are only two ROCs of interest, namely, $\text{ROC}(H(s)) = \{s : Re(s) > -a\}$ and $\text{ROC}(H(s)) = \{s : Re(s) < -a\}$. In the first case, the ROC extends indefinitely to the right side, and the corresponding time-domain signal turns out to be a *causal* signal. In the second case, the ROC extends indefinitely to the left side, and the corresponding time-domain signal turns out to be an *anti-causal* signal.

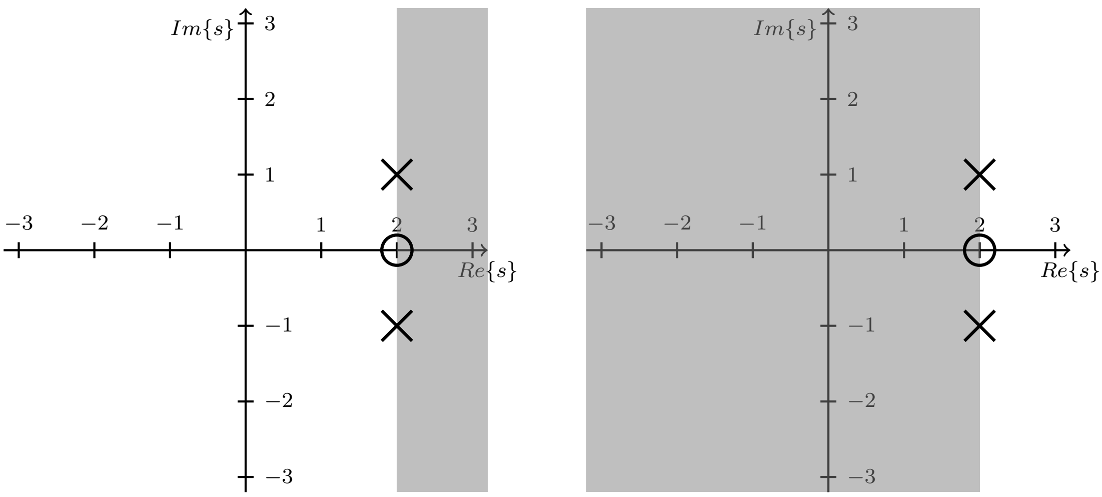
The stability can also directly be read out of the ROC: whenever the ROC *includes the imaginary axis,* then the corresponding time-domain signal is stable (that is, absolutely integrable).



(a) Hence, $h(t) = \begin{cases} 2e^{-2t}\cos(t), & t \geq 0, \\ 0, & t < 0, \end{cases}$
a "stable and causal" signal

(b) Hence, $h(t) = \begin{cases} 0, & t > 0, \\ -2e^{-2t}\cos(t), & t \leq 0, \end{cases}$
an "unstable and anti-causal" signal

Figure 6.3: The two regions of convergence for $H(s) = \frac{2(s+2)}{(s+(2+j))(s+(2-j))}$ (that is, $a = 2$ and $\omega_0 = 1$).



(a) Hence, $h(t) = \begin{cases} 2e^{2t}\cos(t), & t \geq 0, \\ 0, & t < 0, \end{cases}$
an "unstable and causal" signal

(b) Hence, $h(t) = \begin{cases} 0, & t > 0, \\ -2e^{2t}\cos(t), & t \leq 0, \end{cases}$
a "stable and anti-causal" signal

Figure 6.4: The two regions of convergence for $H(s) = \frac{2(s-2)}{(s-(2-j))(s-(2+j))}$ (that is, $a = -2$ and $\omega_0 = 1$).

6.2.3 Key Example 3: Multiple Poles

More generally, in many interesting cases, the Laplace transform is the ratio of two polynomials:

$$H(s) \quad = \quad \frac{P(s)}{Q(s)}, \tag{6.15}$$

where $P(s)$ and $Q(s)$ are polynomials. Not surprisingly, this transfer function again represents several different time-domain signals. How many signals? What types (well-behaved or exploding / right-sided or left-sided)? This is best tackled by looking at the poles in the complex s-plane, and hence, by directly discussing the possible ROCs.

We will study one particular example to gain insight. In particular, let

$$H(s) \quad = \quad \frac{3}{(s+2)(s-1)} = \frac{1}{s-1} - \frac{1}{s+2}. \tag{6.16}$$

Here, we have two poles, one at $s = -2$ and one at $s = 1$. The three possible ROCs are shown in Figure 6.5. Let us now specifically determine the corresponding impulse responses. First, for the case illustrated in Figure 6.5a, we find

$$h(t) \quad = \quad \begin{cases} 0 & t > 0, \\ -e^t + e^{-2t}, & t \leq 0, \end{cases} \tag{6.17}$$

which is anti-causal and unstable (due to the term e^{-2t}). Note again that you can easily verify this result: Simply consider the integral

$$H(s) \quad = \quad \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau, \tag{6.18}$$

and plug in the claimed $h(t)$. It is straightforward to observe that in this case, the integral converges if and only if $Re(s) < -2$, which is exactly the ROC illustrated in Figure 6.5a. In this regime, the integral indeed evaluates to $H(s) = \frac{3}{(s+2)(s-1)}$.

Second, for the case of Figure 6.5b, we find

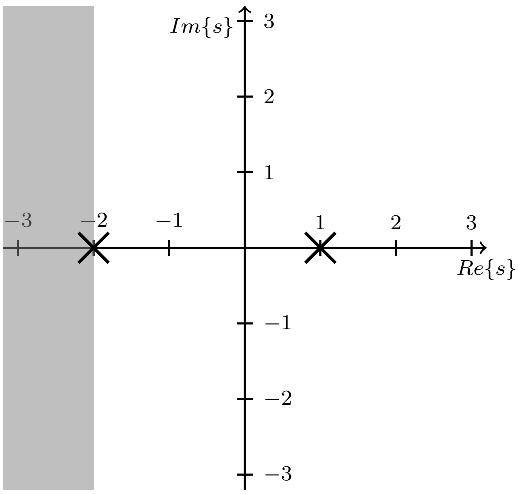
$$h(t) \quad = \quad \begin{cases} e^t - e^{-2t}, & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{6.19}$$

which is causal and unstable (due to the term e^t). Again, the result is easily verified by direct evaluation of the integral.

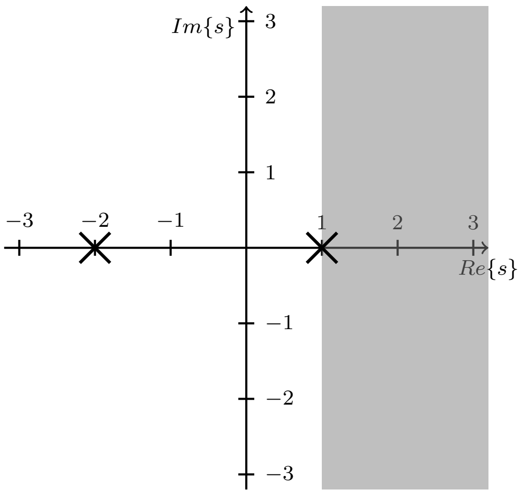
Finally, the most interesting choice of ROC is shown in Figure 6.5c, which is

$$h(t) \quad = \quad \begin{cases} -e^{-2t}, & t \geq 0, \\ -e^t, & t < 0, \end{cases} \tag{6.20}$$

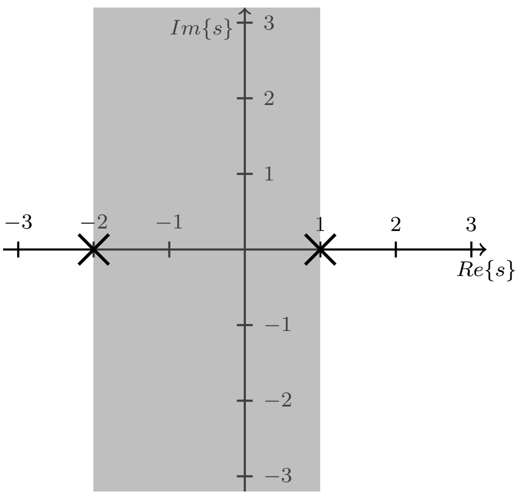
which is a stable system. But note that it is not causal. Rather, its impulse response $h(t)$ is non-zero for all times t. By analogy, the result can be verified by direct evaluation of the integral.



(a) anti-causal and unstable



(b) causal and unstable



(c) two-sided and stable

Figure 6.5: The three possible regions of convergence for two real poles: $H(s) = \frac{3}{(s+2)(s-1)}$.

Summary of the properties of the ROC

When the transfer function is the ratio of two polynomials in s, the first step is to draw out the pole plot (a pole "landscape") in the complex s-plane much like in all the examples we have discussed, except that there will generally be many poles in arbitrary locations. The next step is to identify all possible choices of the ROC. As we have already seen, ROCs must satisfy certain conditions to be valid. To summarize the main requirements:

1. The ROC consists of *strips* parallel to the $j\omega$-axis in the s-plane.
2. The ROC is bounded by poles or extends to infinity. It cannot contain any poles of $H(s)$.
3. If the ROC includes the imaginary axis, then the signal is stable. (This also has an intuitively pleasing additional reason: On the imaginary axis, we have $s = j\omega$, that is, on the imaginary axis, the Laplace transform is exactly equal to the Fourier transform. But as we have already seen, a signal has a Fourier transform if and only if it is stable (i.e., absolutely integrable).)
4. For a causal (i.e., right-sided) signal, the ROC must extend unboundedly to the right (to $Re(s) = \infty$). By analogy, for an anti-causal (i.e., left-sided) signal, the ROC must extend unboundedly to the left (to $Re(s) = -\infty$).

6.3 LTI Systems and The Laplace Transform

Just like in the Fourier domain, we again have the convolution property in the Laplace domain. It should be clear that if we pass a signal of the form of Equation (6.4) through an LTI system with transfer function $H(s)$, the output is simply

$$\begin{aligned} y(t) &= \mathcal{H}\left\{\int X(s)e^{st}ds\right\} \\ &= \int X(s)\mathcal{H}\left\{e^{st}\right\}ds \\ &= \int X(s)H(s)e^{st}ds, \end{aligned} \tag{6.21}$$

where the second equality follows by linearity, and the last equality is Equation (6.2). The key insight from Equation (6.21) is that the Laplace transform $Y(s)$ of the output signal $y(t)$ is simply given by

$$Y(s) = H(s)X(s). \tag{6.22}$$

Again, this insight is also often referred to as the *convolution property* of the Laplace transform, stating that

$$(h * x)(t) \quad \circ\!\!-\!\!\bullet \quad H(s)X(s). \tag{6.23}$$

<table>
<tr><th>$H(s) = \frac{1}{s+a}$</th><th>$a > 0$</th><th>$a < 0$</th></tr>
<tr><td rowspan="2">$\text{ROC}(H(s)) = \{s : Re(s) > -a\}$

that is, interpreted
as a **causal** system</td><td>**stable**</td><td>**unstable**</td></tr>
<tr><td colspan="2">impulse response
$h(t) = \begin{cases} e^{-at}, & t \geq 0, \\ 0, & t < 0. \end{cases}$</td></tr>
<tr><td rowspan="2">$\text{ROC}(H(s)) = \{s : Re(s) < -a\}$

that is, interpreted
as an **anti-causal** system</td><td>**unstable**</td><td>**stable**</td></tr>
<tr><td colspan="2">impulse response
$h(t) = \begin{cases} 0 & t > 0, \\ -e^{-at}, & t \leq 0. \end{cases}$</td></tr>
</table>

Table 6.1: All possible interpretations for a single real-valued pole, that is, for the transfer function $H(s) = \frac{1}{s+a}$, where a is a real number.

Here, we use the shorthand symbol ○—● to mean that on the open circle side is the time-domain representation and on the closed bullet side is the corresponding Laplace domain representation.

Hence, a system can be characterized by its transfer function $H(s)$. But as in our discussion of the Laplace transform, this is not yet enough information. We also have to specify the region of convergence belonging to $H(s)$ in order to have a full characterization of the system. For the simple example of a single pole, i.e., the case of $H(s) = \frac{1}{s+a}$, this is fully described in Table 6.1.

6.4 LTI Systems and The Laplace Transform : Composition

As in the case of the frequency response, the Transfer Function has very desirable properties with respect to certain compositions of LTI systems. The most important are the parallel and the series composition introduced in Section 3.4.

For the parallel composition of LTI systems with transfer functions $H_1(s)$ and $H_2(s)$, respectively, we find an overall transfer function of

$$H(s) \quad = \quad H_1(s) + H_2(s), \tag{6.24}$$

simply by the linearity of the Laplace transform. To understand the properties of the composed system (in terms of causality and stability), it suffices to study the poles of the composed system $H(s)$ (as long as we assume that both $H_1(s)$ and $H_2(s)$ are rational functions of s, as we do throughout this class).

Similarly, for the series composition of LTI systems with transfer functions $H_1(s)$ and $H_2(s)$, respectively, we find an overall transfer function of

$$H(s) \quad = \quad H_1(s)H_2(s), \tag{6.25}$$

which follows from the convolution property of the Laplace transform. Again, to understand the properties of the composed system (in terms of causality and stability), it suffices to study the poles of the composed system $H(s)$.

6.5 LTI Systems and The Laplace Transform : Differential Equations

In many cases, continuous-time LTI systems are conveniently described by differential equations. For those, one could first solve the differential equation assuming that the input is the delta function, thus finding the impulse response $h(t)$. However, there is a much easier approach in this case. To see this, we first observe that if a certain signal $x(t)$ has Laplace transform $X(s)$, then the signal $\frac{d}{dt}x(t)$ must have Laplace transform $sX(s)$. In our notation,

$$\frac{d}{dt}x(t) \quad \circ\!\!-\!\!\bullet \quad sX(s). \tag{6.26}$$

But then, we observe that if two signals $x(t)$ and $y(t)$ satisfy

$$y(t) + 3\frac{d}{dt}y(t) + 2\frac{d^2}{dt^2}y(t) \quad = \quad x(t) + \frac{1}{2}\frac{d}{dt}x(t), \tag{6.27}$$

then their Laplace transforms $X(s)$ and $Y(s)$ must satisfy

$$Y(s) + 3sY(s) + 2s^2Y(s) \quad = \quad X(s) + \frac{1}{2}sX(s). \tag{6.28}$$

First, we trivially rearrange this equation to

$$Y(s)\left(1 + 3s + 2s^2\right) \quad = \quad X(s)\left(1 + \frac{1}{2}s\right). \tag{6.29}$$

To see why this insight is highly valuable, we have to recall from Equation (6.22) that $Y(s) = H(s)X(s)$, which implies that $H(s) = Y(s)/X(s)$. That is, we find

$$H(s) \quad = \quad \frac{1 + \frac{1}{2}s}{1 + 3s + 2s^2}. \tag{6.30}$$

In other words, we found the transfer function of the system in a couple of very easy steps, without solving any differential equations! (If needed, we can now take the inverse Laplace transform of $H(s)$ to recover the impulse response $h(t)$.)

We should also note that this approach does not just work for the example differential equation above; it should be clear that it works for *any* constant-coefficient linear differential equation whatsoever.

6.6 Control Systems : Stability and Causality

The analysis and design of control systems is one of the key applications of the Transfer Function and the Laplace transform. To fully appreciate this is beyond the scope of this class, but we will at least try to give a sense for the issues at hand by the aid of a few examples.

As a first example, let us consider an automobile suspension system. This example is taken from [1, p.473 ff.]. Here, we assume that the road is essentially flat at a reference elevation (no mountains). It has minor deviations from this reference elevation due to defects, potholes, and the like. These deviations are the system input $x(t)$. The suspension system should be designed so that the passengers do not feel the variation in road surface. Therefore, the system output $y(t)$ is the deviation of the chassis position from the desired position above the reference elevation of the road. A classical suspension system has two components: a *spring* (French: *ressort;* German: *Feder*), with some characteristic spring constant k and a *dashpot* (French: *amortisseur;* German: *Stossdämpfer*), again with its own characteristic constant b. A simplified differential equation for the resulting overall system is

$$ky(t) + b\frac{d}{dt}y(t) + M\frac{d^2}{dt^2}y(t) \quad = \quad kx(t) + b\frac{d}{dt}x(t), \tag{6.31}$$

where M is the mass of the vehicle. Note that the main point of our class is *not* to come up with such differential equations — this is what you learn in physics (nevertheless, for this example, we will briefly discuss how to obtain the differential equation). For our class, the main point is the next step: What kind of a system is described by this differential equation, and how is the system behavior influenced by the various parameters in the differential equation?

The Transfer Function and the Laplace transform permit you:

- To analyze and understand the system behavior and the impact of the choice of the parameters k and b.
- To design the overall system, i.e., select k and b in an appropriate fashion.

To see how to do this, we first have to find the transfer function of the system under study. This is a simple matter:

$$H(s) \quad = \quad \frac{k + bs}{Ms^2 + bs + k}. \tag{6.32}$$

This is often referred to as a second-order system since it has two poles. Since second-order systems appear frequently, there is a standard form for them: We express the denominator as $s^2 + 2\zeta\omega_n s + \omega_n^2$. If we do this here, we obtain

$$H(s) \quad = \quad \frac{\omega_n^2 + 2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{6.33}$$

where

$$\omega_n^2 = \frac{k}{M} \quad \text{and} \quad 2\zeta\omega_n = \frac{b}{M}. \tag{6.34}$$

The poles for this system are at

$$\omega_n\left(-\zeta + \sqrt{\zeta^2 - 1}\right) \quad \text{and} \quad \omega_n\left(-\zeta - \sqrt{\zeta^2 - 1}\right). \tag{6.35}$$

Clearly, our system has to be *causal.* This means that the ROC must extend unboundedly to the right (to $Re(s) = \infty$). The question is whether it is stable. As we have seen, for it to be stable, the ROC must include the imaginary axis. In summary, this means that our control system is stable if and only if there are no poles in the right half-plane $Re(s) \geq 0$. Looking back at the formulas for the two poles, it can indeed be verified that the system is always stable as long as ω_n is positive (meaning that b is positive). More precisely, in the interesting regime where $|\zeta| < 1$, we have two complex-conjugate poles in the left half plane (i.e., with $Re(s) < 0$) and thus, the system is stable.

There are further insightful interpretations for this classical second-order system. As one can show:

- the parameter ω_n characterizes how quickly the system responds to a change in the input signal (in our example, this could be a pothole), and
- the parameter ζ, often referred to as the *damping ratio,* characterizes the shape of the response (steep curve of reaction with overshoot versus shallow curve of reaction with no overshoot). This can be interpreted as being related to how close the poles are to the imaginary axis: As we know, if the poles are *on* the imaginary axis, the system is unstable. In extension of this, if the poles are very close to the imaginary axis, the system is "almost unstable," which shows up by a vigorous overshooting. Pictures will be shown in class for illustration.

As a side note, the parameter ω_n is often referred to as the *undamped response frequency* of the system. To understand this terminology, suppose that $\zeta = 0$. Then, the system has two poles on the imaginary axis, located at $j\omega_n$ and $-j\omega_n$. Hence, its impulse response is a cosine/sine of frequency ω_n. Clearly, for the automobile suspension system, this would not be a good choice. (It corresponds essentially to a broken dashpot, and can be observed on certain old cars by leaning on the chassis and then releasing.)

A more general view of control systems is the one captured in Figure 6.6. The idea is that the output of a certain base system (the system $\mathcal{H}$ in the figure) is being measured by a sensor (the system $\mathcal{B}$ in the figure). Ideally, this measurement corresponds to the precise system output, that is, $\tilde{y}(t) = y(t)$. The systems engineer has a reference trajectory $x(t)$ in mind that the system should follow. (For example, the trajectory that a drone is supposed to take.) There is an error $e(t)$ between the actual system output (the actual trajectory of the drone) and the intended trajectory $x(t)$, hence, $e(t) = x(t) - \tilde{y}(t)$. The controller is a cleverly designed system that uses this error signal $e(t)$ in such a way as to generate a system input $z(t)$ that will make the system $\mathcal{H}$ get closer to the desired trajectory.

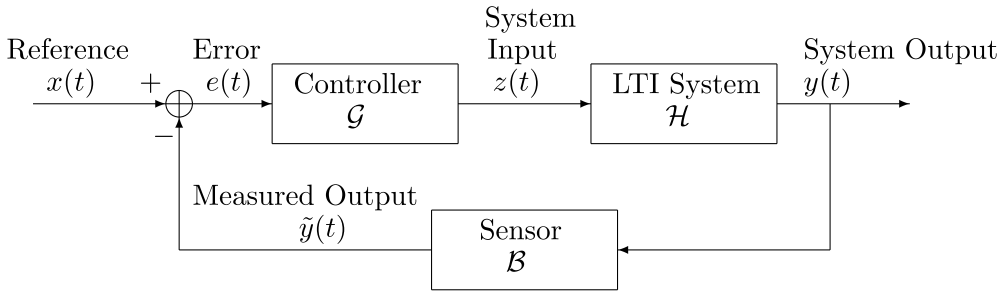


Figure 6.6: A generic view of control systems.

With the tools we have learned so far, we can make a few very easy observations. For example, if we assume that all systems are LTI systems, then we can study transfer functions and Laplace transforms. We immediately make the following two observations:

$$E(s) = X(s) - \tilde{Y}(s) = X(s) - B(s)Y(s) \tag{6.36}$$

$$Y(s) = H(s)G(s)E(s). \tag{6.37}$$

But by plugging in, we obtain

$$Y(s) = H(s)G(s)(X(s) - B(s)Y(s)), \tag{6.38}$$

which we can rearrange into

$$Y(s)(1 + H(s)G(s)B(s)) = H(s)G(s)X(s), \tag{6.39}$$

or more interestingly

$$\frac{Y(s)}{X(s)} = \frac{H(s)G(s)}{1 + H(s)G(s)B(s)}. \tag{6.40}$$

In other words, we have found the end-to-end transfer function of the full system in Figure 6.6:

$$H_{end-to-end}(s) = \frac{H(s)G(s)}{1 + H(s)G(s)B(s)}. \tag{6.41}$$

As earlier, we can now discuss various control strategies (i.e., various systems $G(s)$) in terms of their causality and stability properties. A particularly interesting consideration is how the poles of the end-to-end system move in the s-plane as we change the controller $G(s)$.

Several elements are yet missing from Figure 6.6. Perhaps the most interesting concerns additional external inputs to the system $\mathcal{H}$. For the drone example, this could include the wind and air pressure situation. Often, it is instructive to model these external disturbances as *random processes*. You will encounter models of random processes soon in other classes.

Appendix 6.A Laplace Transform : Properties

Property	Signal	Transform	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
Shift in time	$x(t - t_0)$	$e^{-st_0}X(s)$	R
Shift in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $(s - s_0)$ is in R)
Scaling in time	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	"Scaled" ROC (i.e., s is in the ROC if (s/a) is in R)
Differentiation in time	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Integration in time	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{Re(s) > 0\}$
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Conjugate symmetry	$x(t)$ real-valued	$X(s) = X^*(s^*)$	

Appendix 6.B Laplace Transform : Pairs

	Signal	**Transform**	**ROC**
Dirac delta function	$\delta(t)$ $\delta(t-T)$	1 e^{-sT}	All s All s
Step function	$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise.} \end{cases}$ $-u(-t)$	$\frac{1}{s}$ $\frac{1}{s}$	$Re(s) > 0$ $Re(s) < 0$
	$\frac{t^{n-1}}{(n-1)!}u(t)$ $-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$ $\frac{1}{s^n}$	$Re(s) > 0$ $Re(s) < 0$
One-sided exponential	$e^{-\alpha t}u(t)$ $-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$ $\frac{1}{s+\alpha}$	$Re(s) > -Re(\alpha)$ $Re(s) < -Re(\alpha)$
	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$ $-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$ $\frac{1}{(s+\alpha)^n}$	$Re(s) > -Re(\alpha)$ $Re(s) < -Re(\alpha)$
One-sided Cosines and Sines	$[\cos\omega_0 t]u(t)$ $[\sin\omega_0 t]u(t)$ $[e^{-\alpha t}\cos\omega_0 t]u(t)$ $[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$ $\frac{\omega_0}{s^2+\omega_0^2}$ $\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$ $\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$Re(s) > 0$ $Re(s) > 0$ $Re(s) > -Re(\alpha)$ $Re(s) > -Re(\alpha)$

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