Exercise 1. Let \((S_n, n \in \mathbb{N})\) be the simple symmetric random walk, \((\mathcal{F}_n, n \in \mathbb{N})\) be its natural filtration and 
\[ T = \inf\{n \geq 1 : S_n \geq a \text{ or } S_n \leq -b\}, \]
where \(a, b\) are positive integers.
a) Show that \(T\) is a stopping time with respect to \((\mathcal{F}_n, n \in \mathbb{N})\).
b) Use the optional stopping theorem to compute \(P(\{S_T = a\})\).

Let now \((M_n, n \in \mathbb{N})\) be defined as 
\[ M_n = S_{2n} - n, \quad n \in \mathbb{N}, \]
c) Show that the process \((M_n, n \in \mathbb{N})\) is a martingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\).
d) Apply the optional stopping theorem to compute \(E(T)\).

Remark: Even though \(T\) is an unbounded stopping time, the optional stopping theorem applies both in parts b) and d). Notice that the theorem would not apply if one would consider the following stopping time:
\[ T' = \inf\{n \geq 1 : S_n \geq a\}. \]

Exercise 2. Let \((X_n, n \geq 1)\) be a sequence of i.i.d. random variables such that \(P(\{X_n = +1\}) = p\) and \(P(\{X_n = -1\}) = 1 - p\) for some fixed \(0 < p < 1/2\).

Let \(S_0 = 0\) and \(S_n = X_1 + \ldots + X_n, n \geq 1\). Let also \(\mathcal{F}_0 = \{\emptyset, \Omega\}\) and \(\mathcal{F}_n = \sigma(X_1, \ldots, X_n), n \geq 1\).

Let now \((Y_n, n \in \mathbb{N})\) be the process defined as \(Y_n = \lambda S_n\) for some \(\lambda > 0\) and \(n \in \mathbb{N}\).
a) Using Jensen’s inequality only, for what values of \(\lambda\) can you conclude that the process \(Y\) is a submartingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\)?
b) Identify now the values of \(\lambda > 0\) for which it holds that the process \((Y_n = \lambda S_n, n \in \mathbb{N})\) is a martingale / submartingale / supermartingale with respect to \((\mathcal{F}_n, n \in \mathbb{N})\).
c) Compute \(E(|Y_n|)\) and \(E(Y_n^2)\) for every \(n \in \mathbb{N}\) (and every \(\lambda > 0\)).
d) For what values of \(\lambda > 0\) does it hold that \(\sup_{n \in \mathbb{N}} E(|Y_n|) < +\infty?\) \(\sup_{n \in \mathbb{N}} E(Y_n^2) < +\infty?\)
e) For what values of \(\lambda > 0\) does there exists a random variable \(Y_\infty\) such that \(Y_n \overset{a.s.}{\to} Y_\infty\) as \(n \to \infty\)? Compute the random variable \(Y_\infty\) when it exists (this computation might depend on \(\lambda\)).
f) Finally, for what values of \(\lambda > 0\) does it hold that \(E(Y_\infty | \mathcal{F}_n) = Y_n, \forall n \in \mathbb{N}\)?

Revision exercises. Please have a look at the last 5 exercises of the “100 APA exercises” booklet. The solutions of these will be published when the exam is drawing near (reminder: this one takes place on Saturday, June 25, from 9:15 AM until 12:15 PM, in room CO1).