Exercise 1. Quiz. (18 points) Answer each yes/no question below (1 pt) and provide a short justification for your answer (2 pts).

a) Let $\mathcal{F}$ be a $\sigma$-field on $\Omega$ and $\mathcal{G}, \mathcal{H}$ be two sub-$\sigma$-fields of $\mathcal{F}$.

a1) Does it always hold that $\mathcal{G} \cap \mathcal{H}$ is a $\sigma$-field ?

Note: $\mathcal{G} \cap \mathcal{H}$ is by definition the list of subsets of $\Omega$ belonging to both $\mathcal{G}$ and $\mathcal{H}$.

Answer: Yes. $\emptyset, \Omega \in \mathcal{G} \cap \mathcal{H}$; if $A \in \mathcal{G} \cap \mathcal{H}$, then $A^c \in \mathcal{G} \cap \mathcal{H}$ also; and if $(A_n, n \geq 1)$ is a sequence of events such that $A_n \in \mathcal{G} \cap \mathcal{H}$ for every $n \geq 1$, then $\bigcup_{n \geq 1} A_n \in \mathcal{G}$ and $\bigcup_{n \geq 1} A_n \in \mathcal{H}$, so $\bigcup_{n \geq 1} A_n \in \mathcal{G} \cap \mathcal{H}$.

a2) Does it always hold that $\mathcal{G} \cup \mathcal{H}$ is a $\sigma$-field ?

Note: $\mathcal{G} \cup \mathcal{H}$ is by definition the list of subsets of $\Omega$ belonging to either $\mathcal{G}$ or $\mathcal{H}$.

Answer: No. For example, if $A \in \mathcal{G}$ and $B \in \mathcal{H}$, then it is not always the case that $A \cup B \in \mathcal{G}$ or $A \cup B \in \mathcal{H}$. Example: $\Omega = \{1, 2, 3, 4\}$, $\mathcal{G} = \sigma(\{1, 2\})$, $\mathcal{H} = \sigma(\{1, 3\})$; then $A = \{1, 2\} \in \mathcal{G}$, $B = \{1, 3\} \in \mathcal{G}$, but $A \cup B = \{1, 2, 3\}$ does not belong to $\mathcal{G}$, nor to $\mathcal{H}$.

b) Let $X, Y, Z$ be three random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

b1) Does it always hold that $\sigma(X, Y, Z) = \sigma(X + Y, Y + Z, Z + X)$ ?

Answer: Yes, as the transformation $(x, y, z) \mapsto (x + y, y + z, z + x)$ is a one-to-one transformation (as the computation $\det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 2 \neq 0$ shows). The knowledge of $(X, Y, Z)$ is therefore equivalent to that of $(X + Y, Y + Z, Z + X)$.

b2) If $\sigma(X + Y + Z) = \sigma(X, Y, Z)$, does that necessarily imply that $X, Y, Z$ are independent ?

Answer: No. Here is a simple counter-example: $X = Y = Z$, and $X$ is a non-constant random variable.

c) Let $X$ be a continuous random variable and $Y$ be a discrete random variable which is independent of $X$ and also such that $Y(\omega) \neq 0$ for all $\omega \in \Omega$. Let finally $Z = X \cdot Y$.

c1) Is it always the case that $Z$ is a continuous random variable ?

Answer: Yes. Let $\mathcal{C}$ be the countable set of (non-zero) values of $Y$ and let us compute

\[
F_Z(t) = \sum_{y \in \mathcal{C}} \mathbb{P}(|X \cdot Y | \leq t | Y = y) \mathbb{P}(Y = y)
= \sum_{y \in \mathcal{C}, y > 0} \mathbb{P}(|X \leq t/y|) \mathbb{P}(Y = y) + \sum_{y \in \mathcal{C}, y < 0} \mathbb{P}(|X \geq t/y|) \mathbb{P}(Y = y)
\]
\[ F_Z(t) = \sum_{y \in C, y > 0} F_X(t/y) \mathbb{P}(\{Y = y\}) + \sum_{y \in C, y < 0} (1 - F_X(t/y)) \mathbb{P}(\{Y = y\}) \]

which is differentiable as \( F_X \) is (and \( y \) never takes the value 0).

c2) Assume now that \( X \sim \mathcal{N}(0, 1) \) and \( \mathbb{P}(\{Y = +1\}) = \mathbb{P}(\{Y = +2\}) = \frac{1}{2} \). Is \( Z \) a Gaussian random variable in this case?

**Answer:** No. Using the above formula, we obtain

\[ F_Z(t) = \frac{1}{2} F_X(t) + \frac{1}{2} F_X(t/2) \]

whose derivative is

\[ p_Z(t) = \frac{1}{2\sqrt{2\pi}} \left( \exp(-t^2/2) + \frac{1}{2} \exp(-t^2/8) \right) \]

which is not a Gaussian (it is a centered random variable, but there is no value of \( \sigma > 0 \) such that \( p_Z(t) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp(-t^2/2\sigma^2) \)).

**Exercise 2.** (15 points)

a) Let \( \lambda > 0 \) and \( X \sim \mathcal{E}(\lambda) \), i.e., \( p_X(x) = \lambda e^{-\lambda x} \) for \( x \geq 0 \). What is the distribution of \( Y = \mu X \), where \( \mu > 0 \)?

**Answer:** \( Y \sim \mathcal{E}(\lambda/\mu) \). The simplest way to see this is to note that since \( Y \) is a scaled version of \( X \), it also has an exponential distribution, and that \( \mathbb{E}(Y) = \mu \mathbb{E}(X) = \mu/\lambda \).

b) Let \( X \) be a discrete random variable taking values in \( \mathbb{N} \) and such that \( \mathbb{P}(\{X \geq k\}) = \frac{2}{k(k+1)} \) for every \( k \geq 1 \). Compute \( \mathbb{E}(X) \).

**Answer:** (Side note: observe that \( \mathbb{P}(\{X = 0\}) = 0 \), as \( \mathbb{P}(\{X \geq 1\}) = 1 \). Let us then compute

\[ \mathbb{E}(X) = \sum_{k \geq 1} \mathbb{P}(\{X \geq k\}) = \sum_{k \geq 1} \frac{2}{k(k+1)} = \sum_{k \geq 1} \left( \frac{2}{k} - \frac{2}{k+1} \right) \]

\[ = \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{2}{k} - \frac{2}{k+1} \right) = \lim_{n \to \infty} 2 - \frac{2}{n+1} = 2 \]

Another option is to compute:

\[ \mathbb{E}(X) = \sum_{k \geq 1} k \mathbb{P}(\{X = k\}) = \sum_{k \geq 1} k \left( \mathbb{P}(\{X \geq k\}) - \mathbb{P}(\{X \geq k+1\}) \right) \]

\[ = \sum_{k \geq 1} k \left( \frac{2}{k(k+1)} - \frac{2}{(k+1)(k+2)} \right) = \sum_{k \geq 1} \frac{2k}{k+1} \frac{2}{k+2} \]

\[ = 2 \sum_{k \geq 1} \frac{2}{(k+1)(k+2)} = 2 \sum_{\ell \geq 2} \frac{2}{\ell(\ell+1)} = 2 (\mathbb{E}(X) - 1) \]

so \( \mathbb{E}(X) = 2 \).
c) Let $X$ be a $\mathcal{U}([0,1])$ random variable, $Y$ be independent of $X$ and such that $\mathbb{P}(\{Y = +1\}) = \mathbb{P}(\{Y = -1\}) = \frac{1}{2}$ and $Z = X \cdot Y$. Compute $\phi_Z(t) \in \mathbb{R}$ for $t \in \mathbb{R}$.

**Answer:** Let us compute

$$\phi_Z(t) = \mathbb{E}(\exp(itXY)) = \frac{1}{2} \mathbb{E}(\exp(itX)) + \frac{1}{2} \mathbb{E}(\exp(-itX)) = \frac{1}{2} \frac{e^{it} - 1}{it} + \frac{1}{2} \frac{e^{-it} - 1}{-it} = \frac{e^{it} - e^{-it}}{2it} = \sin(t)$$

Another way to obtain this result is to observe that $Z \sim \mathcal{U}([-1,1])$ and to compute directly the characteristic function.

d) Let $X_1, X_2$ be two square-integrable random variables such that $\text{Var}(X_1 + X_2) = \text{Var}(X_1 - X_2)$. Compute $\text{Cov}(X_1, X_2)$.

**Answer:** Let us expand the variance on both sides:

$$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + 2\text{Cov}(X_1, X_2) + \text{Var}(X_2)$$
$$\text{Var}(X_1 - X_2) = \text{Var}(X_1) - 2\text{Cov}(X_1, X_2) + \text{Var}(X_2)$$

If the 2 variances are equal, then $2\text{Cov}(X_1, X_2) = -2\text{Cov}(X_1, X_2)$, meaning that $\text{Cov}(X_1, X_2) = 0$.

e) Does there exist a non-negative random variable $X$ such that $\mathbb{E}(X) = 10$ and $\mathbb{E}(2^X) = 1000$? Justify your answer.

**Answer:** No. By Jensen’s inequality, it should hold that $2^{\mathbb{E}(X)} \leq \mathbb{E}(2^X)$, but $1024 \not\leq 1000$.

**Exercise 3. (13 points)**

Let $n \geq 1$ and $X_1, \ldots, X_n$ be i.i.d. random variables with common cdf $F(t) = \exp(-\exp(-t))$ for $t \in \mathbb{R}$.

a) Verify that $F$ is indeed a cdf.

**Answer:** $F'(t) = \exp(-\exp(-t)) \exp(-t) > 0$ for all $t \in \mathbb{R}$, so $F$ is increasing; $\lim_{t \to +\infty} F(t) = \exp(-0) = 1$, $\lim_{t \to -\infty} F(t) = \exp(-\infty) = 0$; and $F$ is continuous.

b) Compute both the cdf and the pdf of

$$Y_n = \max\{X_1, \ldots, X_n\} - \ln(n)$$

**Answer:** Let us compute

$$F_{Y_n}(t) = \mathbb{P}(\{\max\{X_1, \ldots, X_n\} \leq t + \ln(n)\}) = \mathbb{P}(\{X_1 \leq t + \ln(n)\})^n = \exp(-\exp(-t - \ln(n)))^n = \exp(-n \exp(-t)/n) = F(t)$$

So for every $n \geq 1$, $Y_n$ has the same cdf as $X_1$, and its pdf is given by

$$p_{Y_n}(t) = \exp(-\exp(-t)) \exp(-t)$$
c) Compute both the cdf and the pdf of
\[ Z_n = \min\{\exp(-X_1), \ldots, \exp(-X_n)\} \]

**Answer:** Two possibilities here: either compute directly

\[ \mathbb{P}(\{Z_n \geq t\}) = \mathbb{P}(\{\exp(-X_1) \geq t\})^n = \mathbb{P}(\{X_1 \leq -\ln(t)\})^n = \exp(-n \exp(\ln(t))) = \exp(-nt) \]

so that \( F_{Z_n}(t) = 1 - \exp(-nt) \) and \( p_{Z_n}(t) = n \exp(-nt) \) [in other words, \( Z_n \sim \mathcal{E}(n) \)].

Or observe that

\[ Z_n = \exp(-\max\{X_1, \ldots, X_n\}) = \exp(-Y_n - \ln(n)) = \frac{\exp(-Y_n)}{n} \]

which by part b) has the same distribution as \( \frac{\exp(-X_1)}{n} \). Then

\[ \mathbb{P}(\{Z_n \leq t\}) = \mathbb{P}(\{\exp(-X_1) \leq nt\}) = \mathbb{P}(\{X_1 \geq -\ln(nt)\}) = 1 - \exp(-nt) \]

reaching the same conclusion.

**Exercise 4. (14 points)**

Let \( \alpha > 0 \) and \((X_n, n \geq 1)\) be a sequence of independent random variables such that

\[ \mathbb{P}(\{X_n = +n^\alpha\}) = \mathbb{P}(\{X_n = -n^\alpha\}) = \frac{1}{2n} \quad \text{and} \quad \mathbb{P}(\{X_n = 0\}) = 1 - \frac{1}{n} \quad \text{for} \ n \geq 1 \]

Let also \( S_n = X_1 + \ldots + X_n \) for \( n \geq 1 \).

a) Compute \( \mathbb{E}(S_n) \) and \( \text{Var}(S_n) \), then estimate both quantities as a function of \( n \) using the approximation (valid for a generic value of \( \gamma \in \mathbb{R} \)):

\[ \sum_{j=1}^{n} j^\gamma \simeq \int_{1}^{n} dx \ x^\gamma \]

(as an example, such an approximation allows to estimate \( \sum_{j=1}^{n} j \simeq \int_{1}^{n} dx \ x \simeq \frac{n^2}{2} \)).

**Answer:** \( \mathbb{E}(S_n) = \sum_{j=1}^{n} \mathbb{E}(X_j) = 0 \) and as the \( X_j \) are independent,

\[ \text{Var}(S_n) = \sum_{j=1}^{n} \text{Var}(X_j) = \sum_{j=1}^{n} \mathbb{E}(X_j^2) = \sum_{j=1}^{n} j^{2\alpha - 1} \]

Using the hint, the variance behaves as \( n \) gets large as

\[ \text{Var}(S_n) \simeq \int_{1}^{n} dx \ x^{2\alpha - 1} \simeq \frac{n^{2\alpha}}{2\alpha} \]
b) For what values of $\beta > 0$ can you show that $\frac{S_n}{n^\beta} \xrightarrow{P} 0$?  

**Answer:** By Chebyshev’s inequality, for every $\varepsilon > 0$,

$$
\mathbb{P}\left(\left\{ \left| \frac{S_n}{n^\beta} \right| \geq \varepsilon \right\} \right) = \mathbb{P}(\{|S_n| \geq n^\beta \varepsilon\}) \leq \frac{\mathbb{E}(S_n^2)}{n^{2\beta} \varepsilon^2} = \frac{\mathbb{Var}(S_n)}{n^{2\beta} \varepsilon^2} \approx \frac{n^{2(\alpha - \beta)}}{2\alpha \varepsilon^2}
$$

which tends to 0 as $n \to \infty$ as soon as $\beta > \alpha$.

c) For what values of $\beta > 0$ can you show that $\frac{S_n}{n^\beta} \xrightarrow{n \to \infty} 0$ almost surely ?

**Answer:** In order to show almost sure convergence via the Borel-Cantelli lemma, we need to show that

$$
\sum_{n \geq 1} \mathbb{P}\left(\left\{ \left| \frac{S_n}{n^\beta} \right| \geq \varepsilon \right\} \right) < +\infty
$$

As

$$
\sum_{n \geq 1} \mathbb{P}\left(\left\{ \left| \frac{S_n}{n^\beta} \right| \geq \varepsilon \right\} \right) \leq \sum_{n \geq 1} \frac{n^{2(\alpha - \beta)}}{2\alpha \varepsilon^2}
$$

the sum is finite if $2(\alpha - \beta) < -1$, i.e., $\beta > \alpha + \frac{1}{2}$.  

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