Homework 9

Exercise 1. Someone proposes you to play the following game: start with an initial amount of $S_0 > 0$ francs, of your choice. Then toss a coin: if it falls on heads, you win $S_0/2$ francs; while if it falls on tails, you lose $S_0/2$ francs. Call $S_1$ your amount after this first coin toss. Then the game goes on, so that your amount after coin toss number $n \geq 1$ is given by

$$S_n = \begin{cases} S_{n-1} + \frac{S_{n-1}}{2} & \text{if coin number } n \text{ falls on heads} \\ S_{n-1} - \frac{S_{n-1}}{2} & \text{if coin number } n \text{ falls on tails} \end{cases}$$

We assume moreover that the coin tosses are independent and fair, i.e., with probability 1/2 to fall on each side. Nevertheless, you should not agree to play such a game: explain why!

Hints:

First, to ease the notation, define $X_n = +1$ if coin $n$ falls on heads and $X_n = -1$ if coin $n$ falls on tails. That way, the above recursive relation may be rewritten as $S_n = S_{n-1} (1 + \frac{X_n}{2})$ for $n \geq 1$.

a) Compute recursively $E(S_n)$; if it were only for expectation, you could still consider playing such a game, but...

b) Define now $Y_n = \log(S_n/S_0)$, and use the central limit theorem to approximate $P(\{Y_n > t\})$ for a fixed value of $t \in \mathbb{R}$ and a relatively large value of $n$. Argue from there why it is definitely not a good idea to play such a game! (computing for example an approximate value of $P(\{S_{100} > S_0/10\})$)

Exercise 2. Let $(G_n, n \geq 1)$ be a sequence of Gaussian random variables with $G_n \sim N(\mu_n, \sigma^2_n)$ for $n \geq 1$.

a) Show, using characteristic functions, that if $\mu_n \to \mu \in \mathbb{R}$ and $\sigma^2_n \to \sigma^2 \geq 0$, then

$$G_n \xrightarrow{d} G \sim N(\mu, \sigma^2)$$

b) Show, still using characteristic functions, that if $\sigma^2_n \to +\infty$, then the sequence $(G_n, n \geq 1)$ does not converge in distribution, irrespective of the sequence $(\mu_n, n \geq 1)$.

Let now $(X_n, n \geq 1)$ be a sequence of independent random variables such that

$$P(\{X_n = +1/\sqrt{n}\}) = P(\{X_n = -1/\sqrt{n}\}) = \frac{1}{2}$$

For $n \geq 1$, let also

$$Y_n = X_1 + \ldots + X_n \quad \text{and} \quad Z_n = X_{n+1} + \ldots + X_{2n}$$

c) Show, using again characteristic functions, that the sequence of random variables $(Z_n, n \geq 1)$ converges in distribution towards a Gaussian random variable $Z$. Compute the variance of this random variable.
Hints: - If \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) is a decreasing function, then it holds that

\[
\sum_{k=n_1+1}^{n_2} f(k) \sim \int_{n_1}^{n_2} dx f(x) \quad \text{as } n_1, n_2 \to \infty
\]

- You will also need to perform some approximations via Taylor expansions.

d*) Show that the sequence of random variables \( (Y_n, n \geq 1) \) does not converge in distribution.

Hint: Consider the subsequence \( (Y_{2k}, k \geq 1) \).