Homework 7

Exercise 1. a) Show that if \((A_n, n \geq 1)\) are independent events in \(F\) and \(\sum_{n \geq 1} P(A_n) = \infty\), then

\[ P\left( \bigcup_{n \geq 1} A_n \right) = 1 \]

Hints: - Start by observing that the statement is equivalent to \(P\left( \bigcap_{n \geq 1} A_n^c \right) = 0\).
- Use the inequality \(1 - x \leq e^{-x}\), valid for all \(x \in \mathbb{R}\).

b) From the same set of assumptions, reach the following stronger conclusion with a little extra effort:

\[ P(\{\omega \in \Omega : \omega \in A_n \text{ infinitely often}\}) = P\left( \bigcap_{N \geq 1} \bigcup_{n \geq N} A_n \right) = 1 \]

which is actually the statement of the second Borel-Cantelli lemma.

c) Let \((X_n, n \geq 1)\) be a sequence of independent random variables such that for some \(\varepsilon > 0\), \(\sum_{n \geq 1} P(\{|X_n| \geq \varepsilon\}) = +\infty\). What can you conclude on the almost sure convergence of the sequence \(X_n\) towards the limiting value 0?

d) Let \((X_n, n \geq 1)\) be a sequence of independent random variables such that \(P(\{X_n = n\}) = p_n = 1 - P(\{X_n = 0\})\) for \(n \geq 1\). What minimal condition on the sequence \((p_n, n \geq 1)\) ensures that

1) \(X_n \xrightarrow{P} 0\) \(n \to \infty\) \(?\) 
2) \(X_n \xrightarrow{L^2} 0\) \(n \to \infty\) \(?\) 
3) \(X_n \xrightarrow{a.s.} 0\) almost surely \(n \to \infty\) \(?\)

e) Let \((Y_n, n \geq 1)\) be a sequence of independent random variables such that \(Y_n \sim \text{Cauchy}(\lambda_n)\) for \(n \geq 1\). What minimal condition on the sequence \((\lambda_n, n \geq 1)\) ensures that

1) \(Y_n \xrightarrow{P} 0\) \(n \to \infty\) \(?\) 
2) \(Y_n \xrightarrow{L^2} 0\) \(n \to \infty\) \(?\) 
3) \(Y_n \xrightarrow{a.s.} 0\) almost surely \(n \to \infty\) \(?\)

Exercise 2. (extended law of large numbers)
Let \((\mu_n, n \geq 1)\) be a sequence of real numbers such that

\[ \lim_{n \to \infty} \frac{\mu_1 + \ldots + \mu_n}{n} = \mu \in \mathbb{R} \]

Let \((X_n, n \geq 1)\) be a sequence of square-integrable random variables such that

\[ E(X_n) = \mu_n, \quad \forall n \geq 1 \quad \text{and} \quad \text{Cov}(X_n, X_m) \leq C_1 \exp(-C_2|m - n|), \quad \forall m, n \geq 1. \]

for some constants \(C_1, C_2 > 0\) (the random variables \(X_n\) are said to be weakly correlated). Let finally \(S_n = X_1 + \ldots + X_n\).

a) Show that

\[ \frac{S_n}{n} \xrightarrow{P} \mu \quad n \to \infty \]
b) Is it also true that
\[ \frac{S_n}{n} \to_{n \to \infty} \mu \] almost surely?
In order to check this, you need to go through the proof of the strong law of large numbers made in class. Does that proof need the fact that the random variables \( X_n \) are independent?

c) Application: Let \((Z_n, n \geq 1)\) be a sequence of i.i.d. \( \sim \mathcal{N}(0, 1) \) random variables, \( x, a \in \mathbb{R} \) and \((X_n, n \geq 1)\) be the sequence of random variables defined recursively as
\[ X_1 = x, \quad X_{n+1} = aX_n + Z_{n+1}, \quad n \geq 1 \]
For what values of \( x, a \in \mathbb{R} \) does the sequence \((X_n, n \geq 1)\) satisfy the assumptions made in a)? Compute \( \mu \) in this case.