Homework 6

Exercise 1. Let X_1, X_2 be two i.i.d. $\mathcal{N}(0,1)$ random variables, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let also

$$Y_1 = |X_1|, \quad Y_2 = |X_2|, \quad R = X_1 + X_2, \quad S = X_1 - X_2 \quad \text{and} \quad T = X_1 \cdot X_2$$

Which of the following assertions are correct? Justify your answer for full credit.

a) $\sigma(X_1, X_2) = \sigma(X_1, X_2, R, S, T)$

b) $\sigma(X_1, X_2) = \sigma(Y_1, Y_2)$

c) $\sigma(X_1, X_2) = \sigma(R, S)$

d) $\sigma(X_1, X_2) = \sigma(Y_1, Y_2, R, T)$

e) $\sigma(X_1, X_2) = \sigma(Y_1, Y_2, S, T)$

Exercise 2. Let $(X_n,\, n \geq 1)$ be independent random variables such that $X_n \sim \text{Bern}(1 - \frac{1}{(n+1)^\alpha})$, where $\alpha > 0$.

Let us also define $Y_n = \prod_{j=1}^n X_j$ for $n \geq 1$.

a) What minimal condition on the parameter $\alpha > 0$ ensures that $Y_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$?

Hint: Use the approximation $1 - x \simeq \exp(-x)$ for x small.

b) Under the same condition as that found in a), does it also hold that $Y_n \underset{n\to\infty}{\overset{L^2}{\to}} 0$?

c) Under the same condition as that found in a), does it also hold that $Y_n \underset{n\to\infty}{\to} 0$ almost surely ?

Hint: If $Y_n = 0$, what can you deduce on Y_m for $m \geq n$?

Exercise 3. a) Show that if $(A_n,\, n \geq 1)$ are *independent* events in $\mathcal{F}$ and $\sum_{n\geq 1} \mathbb{P}(A_n) = \infty$, then

$$\mathbb{P}\Big(\bigcup_{n\geq 1} A_n\Big) = 1$$

Hints: - Start by observing that the statement is equivalent to $\mathbb{P}\left(\bigcap_{n\geq 1} A_n^c\right) = 0$.
- Use the inequality $1 - x \leq e^{-x}$, valid for all $x \in \mathbb{R}$.

b) From the same set of assumptions, reach the following stronger conclusion with a little extra effort:

$$\mathbb{P}(\{\omega \in \Omega : \omega \in A_n \text{ infinitely often}\}) = \mathbb{P}\Big(\bigcap_{N\geq 1} \bigcup_{n\geq N} A_n\Big) = 1$$

which is actually the statement of the *second Borel-Cantelli lemma*.

c) Let $(X_n,\, n \geq 1)$ be a sequence of *independent* random variables such that for some $\varepsilon > 0$, $\sum_{n\geq 1} \mathbb{P}(\{|X_n| \geq \varepsilon\}) = +\infty$. What can you conclude on the almost sure convergence of the sequence X_n towards the limiting value 0?

d) Let $(X_n,\, n \geq 1)$ be a sequence of independent random variables such that $\mathbb{P}(\{X_n = n\}) = p_n = 1 - \mathbb{P}(\{X_n = 0\})$ for $n \geq 1$. What minimal condition on the sequence $(p_n, n \geq 1)$ ensures that

d1) $X_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$? d2) $X_n \underset{n\to\infty}{\overset{L^2}{\to}} 0$? d3) $X_n \underset{n\to\infty}{\to} 0$ almost surely?

e) Let $(Y_n,\, n \geq 1)$ be a sequence of independent random variables such that $Y_n \sim \text{Cauchy}(\lambda_n)$ for $n \geq 1$. What minimal condition on the sequence $(\lambda_n,\, n \geq 1)$ ensures that

e1) $Y_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0$? e2) $Y_n \underset{n\to\infty}{\overset{L^2}{\to}} 0$? e3) $Y_n \underset{n\to\infty}{\to} 0$ almost surely?