Homework 6

Exercise 1. For a generic non-negative random variable $X$ defined on a probability space $(\Omega, \mathcal{F}, P)$, it holds that (the exchange of expectation and integral sign is valid here):

$$E(X) = E\left( \int_0^X dt \right) = E\left( \int_0^{+\infty} 1\{X \geq t\} dt \right) = \int_0^{+\infty} E(1\{X \geq t\}) dt = \int_0^{+\infty} P(\{X \geq t\}) dt$$

a) Use this formula to compute $E(X)$ for $X \sim \mathcal{E}(\lambda)$.

b) Particularize the above formula for $E(X)$ to the case where $X$ takes values in $\mathbb{N}$ only.

c) Use this new formula to compute $E(X)$ for $X \sim \text{Bern}(p)$ and $X \sim \text{Geom}(p)$ for some $0 < p < 1$.

Exercise 2. a) Let $X$ be a square-integrable random variable such that $E(X) = 0$ and $\text{Var}(X) = \sigma^2$. Show that

$$P(\{X \geq t\}) \leq \frac{\sigma^2}{\sigma^2 + t^2} \text{ for } t > 0$$

Hint: You may try various versions of Chebyshev’s inequality here, but not all of them work. A possibility is to use the function $\psi(x) = (x+b)^2$, where $b$ is a free parameter to optimize (but watch out that only some values of $b \in \mathbb{R}$ lead to a function $\psi$ that satisfies the required hypotheses).

b) Let $X$ be a square-integrable random variable such that $E(X) > 0$. Show that

$$P(\{X > t\}) \geq \frac{(E(X) - t)^2}{E(X^2)} \forall 0 \leq t \leq E(X)$$

Hint: Use first Cauchy-Schwarz’ inequality with the random variables $X$ and $Y = 1\{X > t\}$.

Exercise 3. Let $(X_n, n \geq 1)$ be independent random variables such that $X_n \sim \text{Bern}(1 - \frac{1}{(n+1)^\alpha})$, where $\alpha > 0$.

Let us also define $Y_n = \prod_{j=1}^n X_j$ for $n \geq 1$.

a) What minimal condition on the parameter $\alpha > 0$ ensures that $Y_n \xrightarrow{P} 0$ as $n \to \infty$?

Hint: Use the approximation $1 - x \simeq \exp(-x)$ for $x$ small.

b) Under the same condition as that found in a), does it also hold that $Y_n \xrightarrow{L^2} 0$ as $n \to \infty$?

c) Under the same condition as that found in a), does it also hold that $Y_n \xrightarrow{P} 0$ almost surely?

Hint: If $Y_n = 0$, what can you deduce on $Y_m$ for $m \geq n$?