Exercise Set 2 (graded)
Quantum Computation

Exercise 1 *Matrix representation of a few gates / circuits*

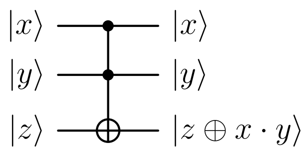
As a reminder, the matrix representation of the CNOT gate in the computational basis $\Big\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\Big\}$ of $\mathbb{C}^4$ (where the control bit is the first one) is given by

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

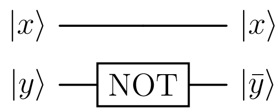
which is a unitary matrix (and also a permutation matrix, with $CNOT^{-1} = CNOT$).

Give the matrix representation in the computational basis of the following four circuits:

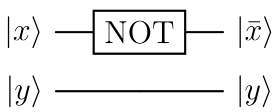
(a)



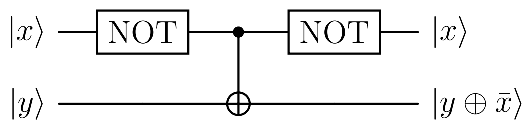
(b)



(c)



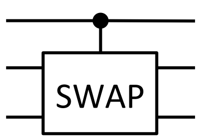
(d)



Are all these matrices also permutation matrices? And are they all equal to their inverse?

Exercise 2 *Fredkin gate*

The SWAP operation takes two input bits and permutes them: SWAP$|b_1, b_2\rangle = |b_2, b_1\rangle$. The Fredkin gate is a three input controlled SWAP gate and is reversible. The gate swaps the two last bits if the first bit is a 1. Otherwise it leaves the input bits unchanged. One intriguing particularity of the Fredkin gate is that it conserves the number of ones.



(a) Show that the irreversible gates AND, OR can be represented in a reversible way from the Fredkin gate.

Hint: Think first how to represent the outputs of the Fredkin gate in the general case.

(b) Give the matrix representation of the Fredkin gate.

(c) Represent the Toffoli (CCNOT) gate in terms of {Fredkin, CNOT}.
Hint: You can achieve this with at most one Fredkin gate and two CNOT gates (and it is helpful to use the matrix representation of these gates).

Exercise 3 *Mach-Zehnder interferometer*

Consider the following matrix product: H(NOT)H.

(a) Is this product unitary ? Why ?

(b) Compute the output for a generic input state $|\varphi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$.

(c) What do you find in the particular cases $|\varphi\rangle = |0\rangle, |1\rangle, |+\rangle, |-\rangle$?

Exercise 4 *Production of Bell states*

(a) Preliminary: which of the following states are product states / entangled states?

(i) $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ (ii) $\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$ (iii) $\frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)$

(iv) $\frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$ (v) $\frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)$

(b) Compute the four Bell states using the following identity using Dirac's notation. Do not use the component and matrix representations.

$$|B_{xy}\rangle = (CNOT)(H \otimes I)(|x\rangle \otimes |y\rangle).$$

where $x, y \in \{0, 1\}$ and $|B_{xy}\rangle$ are the Bell states. *Hint:* $H|x\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}(-1)^x|1\rangle$.

(c) Represent the corresponding circuit.

(d) Represent the circuit corresponding to the inverse identity:

$$|x\rangle \otimes |y\rangle = (H \otimes I)(CNOT)|B_{xy}\rangle$$