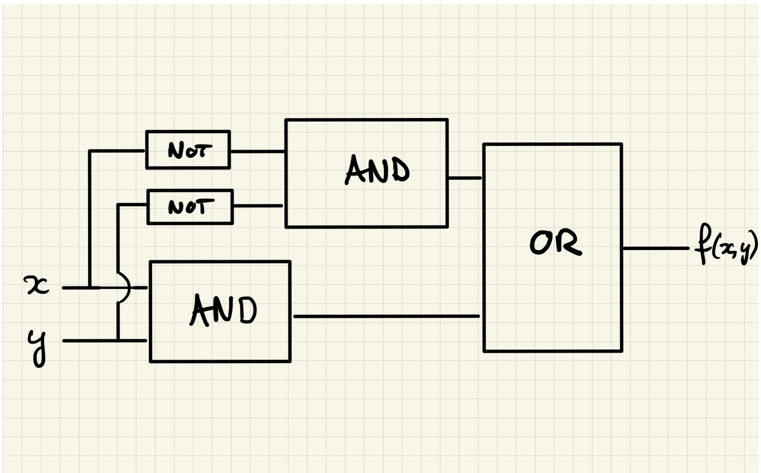
Exercise Set 1 : Solution
Quantum Computation

Exercise 1 *Boolean functions and classical circuits*

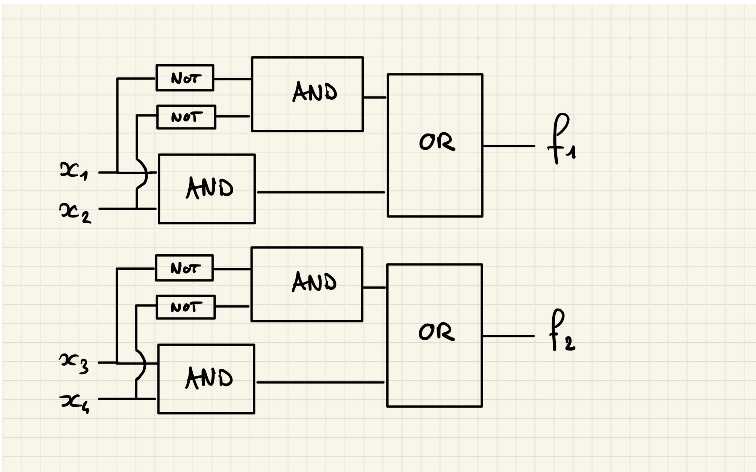
(a) Building a circuit for $f : \{0,1\}^2 \to \{0,1\}$ such that $f(x,y) = 1$ if and only if $x = y$ can be obtained by noting that

$$f(x,y) = 1 \quad \text{if and only if} \quad (x = 1 \text{ and } y = 1) \text{ or } (x = 0 \text{ and } y = 0)$$

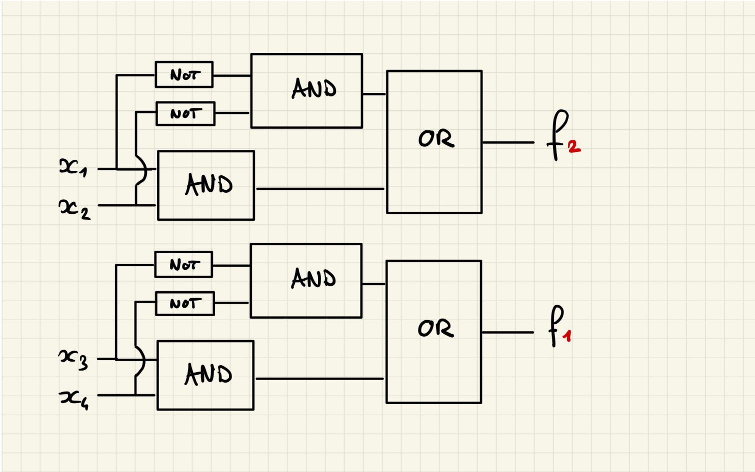
so $f(x,y) = (x \text{ AND } y)$ OR (NOT x AND NOT y) and the circuit is :



The final circuit is given by :

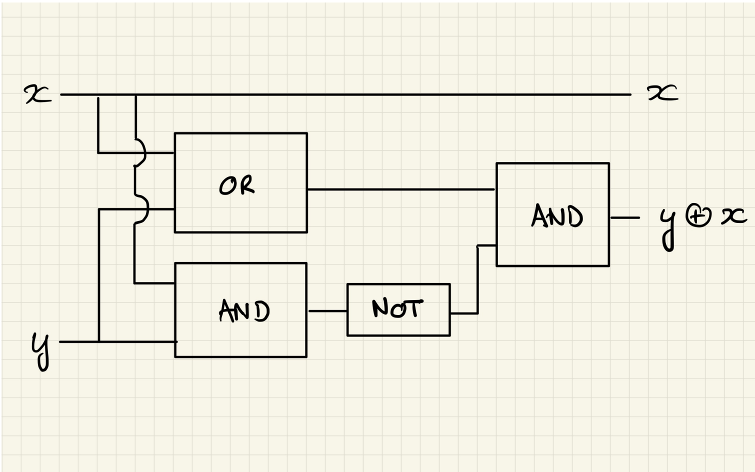


(b) In this case, it suffices to inverts the outputs of f_1 and f_2 in order to obtain what we want :

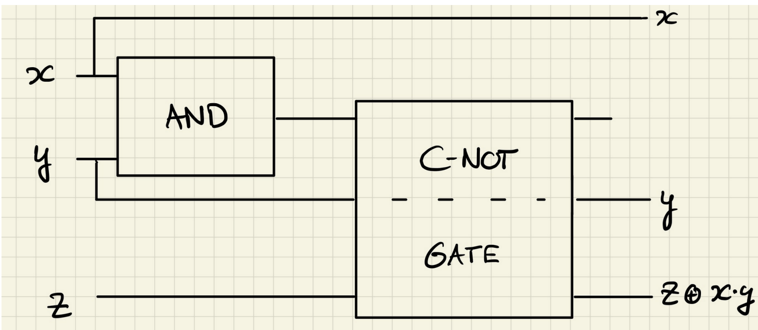


Exercise 2 *NOT, C-NOT, CC-NOT gates*

The C-NOT gate is obtained by noting that $x \oplus y = (x \text{ OR } y)$ AND NOT $(x \text{ AND } y)$:



And here is the CC-NOT or Toffoli gate, obtained via the previous gate :



Exercise 3 *Dirac's notation for vectors and matrices*

(a) If $|w\rangle$ is a vector and α is a scalar, then

$$\big(\alpha\,|w\rangle\,\big)^\dagger = \langle w|\,\overline{\alpha} = \overline{\alpha}\,\langle w|$$

(you can check this in components). Moreover, we have linearity of transposition and complex conjugation :

$$\big(\alpha\,|v\rangle + \beta\,|w\rangle\,\big)^\dagger = \big(\alpha\,|v\rangle\,\big)^\dagger + \big(\beta\,|w\rangle\,\big)^\dagger.$$

(b) Then we get

$$\langle v| = \big(\,|v\rangle\,\big)^\dagger = \big(v_1\,|e_1\rangle + v_2\,|e_2\rangle + \ldots + v_N\,|e_N\rangle\,\big)^\dagger = \overline{v}_1\,\langle e_1| + \overline{v}_2\,\langle e_2| + \ldots + \overline{v}_N\,\langle e_N|\,.$$

(c) If $\langle v| = \sum_{i=1}^N \overline{v}_i\,\langle e_i|$ and $|w\rangle = \sum_{j=1}^N w_j\,|e_j\rangle$, then

$$\langle v|w\rangle = \sum_{i=1}^N\sum_{j=1}^N \overline{v}_i\,w_j\,\,\langle e_i|e_j\rangle \,= \sum_{i=1}^N\sum_{j=1}^N \overline{v}_i\,w_j\,\delta_{ij} = \sum_{i=1}^N \overline{v}_i\,w_i.$$

(d) For $\vec{v} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, we have $\|\vec{v}\|^2 = \vec{v}^\dagger\cdot\vec{v}$, so $\|\vec{v}\|^2 = \overline{\alpha}\,\alpha + \overline{\beta}\,\beta$. On the other hand, $\langle v|v\rangle = \overline{\alpha}\,\alpha + \overline{\beta}\,\beta$ also by (c).

(e) Using components we have :

$$|e_k\rangle\,\langle e_l| = \text{k-th pos}\left\{\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix}\right. \underbrace{\begin{pmatrix}0\cdots 010\cdots 0\end{pmatrix}}_{\text{l-th pos}} = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \vdots & & \vdots & \vdots & \vdots & & \vdots\\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0\\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \vdots & & \vdots & \vdots & \vdots & & \vdots\\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0\end{pmatrix}$$

(f) Thus,

$$A = \sum_{k,l} a_{kl}\,\,|e_k\rangle\,\langle e_l|\,.$$

So,

$$\langle e_i|\,A\,|e_j\rangle = \sum_{k,l} a_{kl}\,\,\langle e_i|e_k\rangle\,\,\langle e_l|e_j\rangle = \sum_{k,l} a_{kl}\,\delta_{ik}\,\delta_{lj} = a_{ij}.$$

(g) From part (e), we have

$$I = \sum_{i=1}^N |e_i\rangle\,\langle e_i|\,.$$

Indeed, $|e_i\rangle\langle e_i|$ is the matrix with 1 at the i-th row and i-th column and zeros elsewhere. This is called the closure relation, and it is valid for any orthonormal basis, as the following computation shows : if $\{|\varphi_i\rangle\}_{i=1\ldots N}$ are orthonormal, there exists a unitary basis change (a "rotation") such that

$$|\varphi_i\rangle = U|e_i\rangle \quad \text{and therefore also} \quad \langle\varphi_i| = \langle e_i| U^\dagger.$$

Then from $I = \sum_{i=1}^N |e_i\rangle\langle e_i|$, we get : $UIU^\dagger = \sum_{i=1}^N U|e_i\rangle\langle e_i|U^\dagger$, so

$$I = \sum_{i=1}^N |\varphi_i\rangle\langle\varphi_i|.$$

(h) From $\alpha_i|\varphi_i\rangle = A|\varphi_i\rangle$, we get directly

$$\sum_{i=1}^N \alpha_i|\varphi_i\rangle\langle\varphi_i| = \sum_{i=1}^N A|\varphi_i\rangle\langle\varphi_i| = A\sum_{i=1}^N |\varphi_i\rangle\langle\varphi_i| = A\,I = A.$$

Exercise 4 *Tensor Product in Dirac's notation*

(a) By distributivity of the tensor product (first two properties), it follows that :

$$|v\rangle_1 \otimes |w\rangle_2 = \left(\sum_{i=1}^N v_i|e_i\rangle_1\right) \otimes \left(\sum_{j=1}^M w_j|f_j\rangle_2\right) = \sum_{i=1}^N\sum_{j=1}^M v_i w_j |e_i\rangle_1 \otimes |f_j\rangle_2.$$

(b) Take two vectors $|e_i, f_j\rangle$ and $|e_k, f_l\rangle$ of $\mathcal{H}_1 \bigotimes \mathcal{H}_2$. Then by definition of the inner product :

$$\langle e_k, f_l|e_i, f_j\rangle = \langle e_k|e_i\rangle \cdot \langle f_l|f_j\rangle = \delta_{ki}\cdot\delta_{lj}.$$

So this equals one if and only if $(k,l) = (i,j)$ and zero otherwise. This means that $\big\{\,|e_i, f_j\rangle\,; i = 1\ldots N; j = 1\ldots M\big\}$ is an orthonormal basis of $\mathcal{H}_1 \bigotimes \mathcal{H}_2$. The dimension equals the number of basis vectors, so is $N\cdot M$, the product of $\dim\mathcal{H}_1$ and $\dim\mathcal{H}_2$.

(c) We apply the definition

$$A\otimes B\,|\Psi\rangle = \sum_{i,j}\psi_{ij}A|e_i\rangle_1 \otimes B|f_j\rangle_2$$

to $|\Psi\rangle = |e_k, f_l\rangle$. So $\psi_{ij} = 1$ for $(i,j) = (k,l)$ and 0 otherwise. This means :

$$A\otimes B\,|e_k, f_l\rangle = A|e_k\rangle \otimes B|f_l\rangle$$

and multiplying by $\langle e_i, f_j|$, we find :

$$\begin{aligned}\langle e_i, f_j|\,A\otimes B\,|e_k, f_l\rangle &= \big(\langle e_i|\otimes\langle f_j|\big)\big(A|e_k\rangle\otimes B|f_l\rangle\big)\\ &= \langle e_i|A|e_k\rangle\langle f_j|B|f_l\rangle = a_{ik}b_{jl}.\end{aligned}$$

(d) The formulas follow by translating the formulas found in (a) and (c) to the component notation.