Exercise Set 1
Quantum Computation

**Exercise 1** *Boolean functions and classical circuits*

(a) Build a classical circuit that computes the Boolean function $f : \{0,1\}^4 \to \{0,1\}^2$ defined as follows :

$$f(x_1, x_2, x_3, x_4) = \begin{cases} (1,1) & \text{if } x_1 = x_2 \text{ and } x_3 = x_4 \\ (1,0) & \text{if } x_1 = x_2 \text{ and } x_3 \neq x_4 \\ (0,1) & \text{if } x_1 \neq x_2 \text{ and } x_3 = x_4 \\ (0,0) & \text{if } x_1 \neq x_2 \text{ and } x_3 \neq x_4 \end{cases}$$

*Indication :* Start by building a circuit for $f : \{0,1\}^2 \to \{0,1\}$ such that $f(x,y) = 1$ if and only if $x = y$ : this circuit can then be used as a sub-circuit of the desired circuit.

(b) More difficult ( ?) case : build a circuit for

$$f(x_1, x_2, x_3, x_4) = \begin{cases} (1,1) & \text{if } x_1 = x_2 \text{ and } x_3 = x_4 \\ (0,1) & \text{if } x_1 = x_2 \text{ and } x_3 \neq x_4 \\ (1,0) & \text{if } x_1 \neq x_2 \text{ and } x_3 = x_4 \\ (0,0) & \text{if } x_1 \neq x_2 \text{ and } x_3 \neq x_4 \end{cases}$$

**Exercise 2** *NOT, C-NOT, CC-NOT gates*

In the course, we have shown that {NOT, C-NOT, CC-NOT} is a universal set of gates, as these gates can be used to build each gate in the set {AND, OR, NOT, COPY}, which is a universal set of gates, according to Post's theorem.

In this exercise, we ask you to prove the reverse statement, namely, that each gate from the set {NOT, C-NOT, CC-NOT} can be constructed with gates in the set {AND, OR, NOT, COPY}.

**Exercise 3** *Dirac's notation for vectors and matrices*

Let $\mathcal{H} = \mathbb{C}^N$ be a vector space of $N$ dimensional vectors with complex components. Let

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

be a column vector. We define its "conjugate" as

$$\vec{v}^\dagger = (\overline{v}_1, \dots, \overline{v}_N)$$

where $\overline{z}$ stands for the complex conjugate of $z \in \mathbb{C}$. So $\vec{v}^\dagger$ is obtained by transposition and complex conjugation (if the components are real this reduces to the usual transposed vector). The inner or scalar product is by definition

$$\vec{v}^\dagger \cdot \vec{w} = \overline{v}_1\, w_1 + \ldots + \overline{v}_N\, w_N$$

and the norm is

$$\|\vec{v}\|^2 = \vec{v}^\dagger \cdot \vec{v} = \overline{v}_1\, v_1 + \ldots + \overline{v}_N\, v_N = |v_1|^2 + \ldots + |v_N|^2$$

In Dirac's notation we write $\vec{v} = |v\rangle$ and $\vec{v}^\dagger = \langle v|$. Therefore the inner product becomes

$$\langle v|w\rangle = \overline{v}_1\, w_1 + \ldots + \overline{v}_N\, w_N$$

The canonical orthonormal basis vectors are by definition

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \vec{e}_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

In Dirac's notation the orthonormality of the basis vectors is expressed as

$$\langle e_i|e_j\rangle = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

The expansion of a vector on this basis is

$$|v\rangle = v_1\,|e_1\rangle + v_2\,|e_2\rangle + \ldots + v_N\,|e_N\rangle$$

Now you will check a few easy facts of linear algebra and translate them in Dirac's notation.

(a) Check from the definitions above that if $|v\rangle = \alpha\,|v'\rangle + \beta\,|v''\rangle$ then

$$\langle v| = \overline{\alpha}\,\langle v'| + \overline{\beta}\,\langle v''|\,.$$

(b) In particular deduce that if $|v\rangle = v_1 |e_1\rangle + v_2 |e_2\rangle + \ldots + v_N |e_N\rangle$ then

$$\langle v| = \overline{v}_1 \ \langle e_1| + \overline{v}_2 \ \langle e_2| + \ldots + \overline{v}_N \ \langle e_N| \, .$$

(c) Show directly in Dirac notation that

$$\langle v|w\rangle = \overline{v}_1 \, w_1 + \ldots + \overline{v}_N \, w_N.$$

(d) Deduce that $\sqrt{\langle v|v\rangle} = \|v\|$.

(e) Consider the ket-bra expression $|e_i\rangle \langle e_j|$ for canonical basis vectors. Write this as an $N \times N$ matrix.

(f) Consider now an $N \times N$ matrix $A$ with complex matrix elements $a_{ij}$ ; $i = 1 \ldots N$ ; $j = 1 \ldots N$. Deduce from the above question that

$$A = \sum_{i,j=1}^{N} a_{ij} \, |e_i\rangle \langle e_j|$$

and also that

$$a_{ij} = \langle e_i| \, A \, |e_j\rangle \, .$$

(g) In particular verify that the identity matrix satisfies :

$$I = \sum_{i=1}^{N} |e_i\rangle \langle e_i| \, .$$

This is called the closure relation. Deduce that in fact this relation is valid for any orthonormal basis of vectors $|\varphi_i\rangle$, $i = 1, \ldots, N$.

(h) (Spectral theorem) Let $A = A^\dagger$ where $A^\dagger = A^{T,*}$. This is called a *Hermitian matrix.* An important theorem of linear algebra states that : "any Hermitian matrix has $N$ orthonormal eigenvectors with real eigenvalues (possibly degenerate)". Let $|\varphi_i\rangle \, , \alpha_i$, $i = 1, \ldots, N$ be the eigenvectors and eigenvalues of $A$, *i.e.*,

$$A \, |\varphi_i\rangle = \alpha_i \, |\varphi_i\rangle \, .$$

Prove direclty in Dirac's notation that

$$A = \sum_{i=1}^{N} \alpha_i \, |\varphi_i\rangle \langle \varphi_i| \, .$$

This "expansion" is often called the spectral expansion (or theorem).

**Exercise 4** *Tensor Product in Dirac's notation*

Let $\mathcal{H}_1 = \mathbb{C}^N$ and $\mathcal{H}_2 = \mathbb{C}^M$ be $N$ and $M$ dimensional Hilbert spaces. The tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a new Hilbert space formed by "pairs of vectors" denoted as $|v\rangle_1 \otimes |w\rangle_2 \equiv |v, w\rangle$ with the properties :

- $\big(\alpha |v\rangle_1 + \beta |v'\rangle_1\big) \otimes |w\rangle_2 = \alpha |v\rangle_1 \otimes |w\rangle_2 + \beta |v'\rangle_1 \otimes |w\rangle_2$,
- $|v\rangle_1 \otimes \big(\alpha |w\rangle_2 + \beta |w'\rangle_2\big) = \alpha |v\rangle_1 \otimes |w\rangle_2 + \beta |v\rangle_1 \otimes |w'\rangle_2$,
- $\big(|v\rangle_1 \otimes |w\rangle_2\big)^\dagger = \langle v|_1 \otimes \langle w|_2$,
- $\langle v, w|v', w'\rangle = \langle v|v'\rangle_1 \langle w|w'\rangle_2$.

(a) Show that for any two vectors of $\mathcal{H}_1$ and $\mathcal{H}_2$ expanded on two basis, $|v\rangle_1 = \sum_{i=1}^N v_i |e_i\rangle_1$ and $|w\rangle_2 = \sum_{j=1}^M w_j |f_j\rangle_2$, then

$$|v\rangle_1 \otimes |w\rangle_2 = \sum_{i=1}^N \sum_{j=1}^M v_i w_j |e_i\rangle_1 \otimes |f_j\rangle_2 .$$

(b) Show that if $\{|e_i\rangle_1 ; i = 1 \dots N\}$ and $\{|f_j\rangle_2 ; j = 1 \dots M\}$ are orthonormal, then $|e_i\rangle_1 \otimes |f_j\rangle_2 \equiv |e_i, f_j\rangle$ is an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$. What is the dimension of $\mathcal{H}_1 \otimes \mathcal{H}_2$ ?

(c) Any vector $|\Psi\rangle$ of $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be expanded on the basis $|e_i\rangle_1 \otimes |f_j\rangle_2 \equiv |e_i, f_j\rangle, i = 1 \dots N, j = 1 \dots M$,

$$|\Psi\rangle = \sum_{i=1, j=1}^{N,M} \psi_{ij} |e_i, f_j\rangle .$$

If $A$ is a matrix acting on $\mathcal{H}_1$ and $B$ is a matrix acting on $\mathcal{H}_2$, the tensor product $A \otimes B$ is defined as

$$A \otimes B |\Psi\rangle = \sum_{i,j} \psi_{ij} A |e_i\rangle_1 \otimes B |f_j\rangle_2 .$$

Check that the matrix elements of $A \otimes B$ in the basis $|e_i, f_j\rangle$ are :

$$\langle e_i, f_j| A \otimes B |e_k, f_l\rangle = a_{ik} b_{jl}.$$

(d) Let $\mathcal{H}_1 = \mathbb{C}^2$, $\mathcal{H}_2 = \mathbb{C}^2$. Take $A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, $|v\rangle_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $|w\rangle_2 = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$. From the defining properties of the tensor product deduce the the following formulas :

$$|v\rangle_1 \otimes |w\rangle_2 = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}, \qquad A_1 \otimes B_2 = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}.$$

These are often useful in order to do calculations in components.