Solutions to Homework 5

Exercise 1.

a) Option 1: by the assumptions made, $\text{Cov}(X_1 + X_2, X_1 - X_2) = \text{Var}(X_1) + \text{Cov}(X_2, X_1) - \text{Cov}(X_1, X_2) - \text{Var}(X_2) = \text{Var}(X_1) - \text{Var}(X_2) = 0$. Besides, as X_1, X_2 are independent Gaussian random variables, $X = (X_1, X_2)$ is a Gaussian random vector, so $(X_1 + X_2, X_1 - X_2)$ is also a Gaussian random vector whose components are uncorrelated, and therefore independent, by Proposition 6.8 of the course.

Option 2 is to show directly that

$$\mathbb{E}(e^{it_1(X_1+X_2)+it_2(X_1-X_2)}) = \mathbb{E}(e^{it_1(X_1+X_2)})\,\mathbb{E}(e^{it_2(X_1-X_2)}) \quad \forall t_1, t_2 \in \mathbb{R}$$

as this would imply independence of $X_1 + X_2$ and $X_1 - X_2$. We check indeed that

$$\begin{aligned}&\mathbb{E}(e^{it_1(X_1+X_2)+it_2(X_1-X_2)}) = \mathbb{E}(e^{i(t_1+t_2)X_1+i(t_1-t_2)X_2)}) \\ &= \mathbb{E}(e^{i(t_1+t_2)X_1})\,\mathbb{E}(e^{i(t_1-t_2)X_2}) = e^{i\mu_1(t_1+t_2)-\sigma_1^2(t_1+t_2)^2/2}\, e^{i\mu_2(t_1-t_2)-\sigma_2^2(t_1-t_2)^2/2}\end{aligned}$$

Because of the assumption made ($\sigma_1^2 = \sigma_2^2 = \sigma^2$), the above expression is further equal to

$$\begin{aligned}&= e^{i(\mu_1+\mu_2)t_1+i(\mu_1-\mu_2)t_2-\sigma^2(t_1^2+t_2^2)} = e^{i(\mu_1+\mu_2)t_1-\sigma^2 t_1^2}\, e^{i(\mu_1-\mu_2)t_2-\sigma^2 t_2^2} \\ &= \mathbb{E}(e^{it_1(X_1+X_2)})\,\mathbb{E}(e^{it_2(X_1-X_2)})\end{aligned}$$

which proves the claim.

b) 1. Skipped. Just note that closing our eyes, we could compute

$$\phi_X'(t) = i\,\mathbb{E}(X\, e^{itX}) \quad \text{and} \quad \phi_X''(t) = -\mathbb{E}(X^2 e^{itX}), \quad t \in \mathbb{R}$$

and deduce from there that indeed, if $\mathbb{E}(X^2) < +\infty$, then ϕ_X is twice continuously differentiable. As a by-product, we obtain the relation

$$\phi_X''(0) = -\mathbb{E}(X^2)$$

from the second formula evaluated in $t = 0$.

2. Skipped.

3. By the assumptions made, we obtain

$$\mathbb{E}(e^{it_1(X_1+X_2)+it_2(X_1-X_2)}) = \mathbb{E}(e^{it_1(X_1+X_2)})\,\mathbb{E}(e^{it_2(X_1-X_2)})$$

and also

$$\mathbb{E}(e^{it_1(X_1+X_2)+it_2(X_1-X_2)}) = \mathbb{E}(e^{i(t_1+t_2)X_1+i(t_1-t_2)X_2}) = \phi_{X_1}(t_1+t_2)\,\phi_{X_2}(t_1-t_2)$$

so

$$\log \phi_{X_1}(t_1+t_2) + \log \phi_{X_2}(t_1-t_2) = \log \mathbb{E}(e^{it_1(X_1+X_2)}) + \log \mathbb{E}(e^{it_2(X_1-X_2)}) = g_1(t_1) + g_2(t_2)$$

proving the claim.

4. Differentiating first the equality with respect to t_1, we obtain

$$f_1'(t_1+t_2) + f_2'(t_1-t_2) = g_1'(t_1)$$

and then with respect to t_2:

$$f_1''(t_1+t_2) - f_2''(t_1-t_2) = 0$$

Setting $t_1 = t_2 = \frac{t}{2}$ leads to $f_1''(t) = f_2''(0)$, and setting $t_1 = -t_2 = \frac{t}{2}$ leads to $f_2''(t) = f_1''(0)$. As these equalities are satisfied for arbitrary $t \in \mathbb{R}$, this says that the second derivatives of both f_1 and f_2 are constant functions, therefore that both f_1 and f_2 are polynomials of degree less than or equal to 2.

5. The assumption is that $\log \phi_X(t) = at^2 + bt + c$ for $t \in \mathbb{R}$. Using the hint and writing $\mu = \mathbb{E}(X)$, $\sigma^2 = \mathrm{Var}(X)$, we obtain successively:

$$\begin{aligned} e^c = \phi_X(0) = 1 & \quad \text{so } c = 0 \\ b = \phi_X'(0) = i\mu & \quad \text{so } b = i\mu \\ 2a + b^2 = \phi_X''(0) = -\mathbb{E}(X^2) = -(\mu^2+\sigma^2) & \quad \text{so } a = -\sigma^2/2 \end{aligned}$$

Therefore, $\phi_X(t) = e^{i\mu t - \sigma^2 t^2/2}$, which is the characteristic function of a Gaussian.

6. As X_1, X_2 are independent and Gaussian, this implies that (X_1, X_2) is a Gaussian vector, i.e., that X_1, X_2 are jointly Gaussian. By the assumptions made, we also have

$$0 = \mathrm{Cov}(X_1+X_2, X_1-X_2) = \mathrm{Var}(X_1) + \mathrm{Cov}(X_2, X_1) - \mathrm{Cov}(X_1, X_2) - \mathrm{Var}(X_2) = \mathrm{Var}(X_1) - \mathrm{Var}(X_2)$$

so $\mathrm{Var}(X_1) = \mathrm{Var}(X_2)$ [note in passing that we did not use here the assumption that X_1 and X_2 are uncorrelated]. This finally completes the proof of the result stated in part b).

Exercise 2.

a) The result follows directly from the Chebyshev-Markov inequality with $\psi(x) = e^{tx}$.

b) We can write $X = \sum_{i=1}^n B_i$, where the B_i's are n iid Bernoulli(p) random variables. Then, for each B_i we have

$$\mathbb{E}(e^{tB_i}) = pe^t + 1 - p$$

so that we have

$$\begin{aligned} M_X(t) &= \mathbb{E}(e^{tX}) \\ &= \mathbb{E}(e^{t\sum B_i}) \\ &= \mathbb{E}\left(\prod_i e^{tB_i}\right) \\ &= \prod_i \mathbb{E}(e^{tB_i}) \\ &= (pe^t + 1 - p)^n. \end{aligned}$$

c) By applying the inequality in part 1 to X with $a = qn$, we get

$$\mathbb{P}(X \geq gn) \leq \left(\frac{pe^t + 1 - p}{e^{tq}}\right)^n$$

Since y^n is an increasing function for $y > 0$, in order to optimize the right-hand side over t, we can substitute $z = e^t$ and optimize the function

$$\frac{pz + 1 - p}{z^q}$$

over $z > 0$. By taking the derivative and putting it equal to 0, we get

$$\frac{pz^q - qz^{q-1}(pz + 1 - p)}{z^{2q}} = 0 \iff pz - pqz - q(1-p) = 0 \iff z = \frac{q}{p} \cdot \frac{1-p}{1-q}.$$

Substituting $z = e^t$ in the right-hand side of the inequality leads to the result.

d) We have that

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_i B_i\right) = \sum_i \mathbb{E}(B_i) = np$$

so that Markov inequality for $a = qn$ becomes

$$\mathbb{P}(X \geq qn) \leq \frac{\mathbb{E}(X)}{nq} = \frac{np}{nq} = \frac{p}{q}.$$

Note that the second inequality does not depend on n. This is in general bad. In fact, when n is large we expect X to concentrate around np (its expectation). Since $q > p$, we therefore expect that $\mathbb{P}(X \geq qn) \to 0$ when $n \to \infty$. This is indeed what we get from the first inequality: the right-hand side goes to 0 when $n \to \infty$. However, the second inequality is just a constant for every n, and therefore it is very loose when n is large.Therefore, the inequality from part-(c) is better.

Exercise 3.

a) Using $\psi(x) = x^2$ or $\psi(x) = \sigma^2 + x^2$ in Chebyshev's inequality leads to respectively

$$\mathbb{P}(\{X \geq t\}) \leq \frac{\sigma^2}{t^2} \quad \text{and} \quad \mathbb{P}(\{X \geq t\}) \leq \frac{2\sigma^2}{\sigma^2 + t^2}$$

which is not what we want. Using the hint (with $b \geq 0$ in order to satisfy the hypotheses), we obtain

$$\mathbb{P}(\{X \geq t\}) \leq \frac{\mathbb{E}((X+b)^2)}{(t+b)^2} = \frac{\sigma^2 + b^2}{(t+b)^2}$$

Optimizing over the parameter b, we find that best possible bound is obtained by setting $b^* = \frac{\sigma^2}{t}$ (which is non-negative), leading to

$$\mathbb{P}(\{X \geq t\}) \leq \frac{\sigma^2}{\sigma^2 + t^2}$$

b) Using Cauchy-Schwarz's inequality with the random variables X and $Y = 1_{\{X>t\}}$, we obtain

$$\mathbb{E}(X\, 1_{\{X>t\}})^2 \leq \mathbb{E}(X^2)\, \mathbb{P}(\{X > t\})$$

On the other hand, we have $\mathbb{E}(X\, 1_{\{X>t\}}) = \mathbb{E}(X) - \mathbb{E}(X\, 1_{\{X\leq t\}}) \geq \mathbb{E}(X) - t$, therefore the result.