Solutions to Homework 5

Exercise 1. a) True; Note that for any random variable $X$, and a function of $X$, say $f(X)$ we have $\sigma(f(X)) = \sigma(X)$. Thus, $\sigma(X_1, X_2) = \sigma(X_1, X_2, R, S, T)$ as $R, S, T$ are functions of $X_1, X_2$.

b) False; For any random variable $X$ and a surjective function of $X$, say $f(X)$ we have that $\sigma(f(X)) \not\subseteq \sigma(X)$. Here, the absolute value function, $\cdot |$, is not injective thus information about the sign of the random variables $X_1, X_2$ is lost in $Y_1, Y_2$. Thus, $\sigma(Y_1, Y_2) \not\subseteq \sigma(X_1, X_2)$.

c) True; Note that $R = X_1 + X_2$ and $S = X_1 - X_2$. Thus, $\sigma(R, S) = \sigma(R + S, R - S)$ as the mapping from $(R, S) \rightarrow (R + S, R - S)$ is a bijection. Further, $\sigma(R + S, R - S) = \sigma(X, Y)$.

d) False; Consider the case where $R = 0$ which implies $T \leq 0$. Thus $X_1 = -X_2$, and having the additional information about the absolute values of $X_1$ and $X_2$ (i.e., $Y_1$ and $Y_2$, respectively) doesn’t provide any information about the signs of the random variables $X_1$ and $X_2$.

e) False; We will follow the same approach as in the part (d) to see if we can reconstruct information about $X_1$ and $X_2$ from $Y_1, Y_2, S$ and $T$. Consider the case where $S = 0$ which also implies $T \geq 0$. Thus, $X_1 = X_2$. However, even after having the additional information about $Y_1$ and $Y_2$, it doesn’t tell us whether the values taken by $X_1$ and $X_2$ are positive or negative.

Exercise 2. a) Option 1: by the assumptions made, $\text{Cov}(X_1 + X_2, X_1 - X_2) = \text{Var}(X_1) + \text{Cov}(X_2, X_1) - \text{Cov}(X_1, X_2) - \text{Var}(X_2) = \text{Var}(X_1) - \text{Var}(X_2) = 0$. Besides, as $X_1, X_2$ are independent Gaussian random variables, $X = (X_1, X_2)$ is a Gaussian random vector, so $(X_1 + X_2, X_1 - X_2)$ is also a Gaussian random vector whose components are uncorrelated, and therefore independent, by Proposition 6.8 of the course.

Option 2 is to show directly that

$$\mathbb{E}(e^{it_1(X_1+X_2)+it_2(X_1-X_2)}) = \mathbb{E}(e^{it_1(X_1+X_2)}) \mathbb{E}(e^{it_2(X_1-X_2)}) \forall t_1, t_2 \in \mathbb{R}$$

as this would imply independence of $X_1 + X_2$ and $X_1 - X_2$. We check indeed that

$$\mathbb{E}(e^{it_1(X_1+X_2)+it_2(X_1-X_2)}) = \mathbb{E}(e^{it_1(t_1+t_2)X_1+it_1(t_1-t_2)X_2})$$

$$= \mathbb{E}(e^{it_1(t_1-t_2)X_1}) \mathbb{E}(e^{it_1(t_1-t_2)X_2}) = e^{it_1 \mu_1(t_1+t_2)-\sigma_1^2(t_1+t_2)^2/2} e^{it_1 \mu_2(t_1-t_2)-\sigma_2^2(t_1-t_2)^2/2}$$

Because of the assumption made ($\sigma_1^2 = \sigma_2^2 = \sigma^2$), the above expression is further equal to

$$= e^{it_1(\mu_1+\mu_2)t_1+i(\mu_1-\mu_2)t_2-\sigma_1^2t_1^2/2} e^{it_2(\mu_1-\mu_2)t_2-\sigma_2^2t_2^2/2}$$

$$= \mathbb{E}(e^{it_1(X_1+X_2)}) \mathbb{E}(e^{it_2(X_1-X_2)})$$

which proves the claim.

b) 1. Skipped. Just note that closing our eyes, we could compute

$$\phi_X(t) = i \mathbb{E}(X e^{itX}) \quad \text{and} \quad \phi_X''(t) = -\mathbb{E}(X^2 e^{itX}), \quad t \in \mathbb{R}$$

and deduce from there that indeed, if $\mathbb{E}(X^2) < +\infty$, then $\phi_X$ is twice continuously differentiable. As a by-product, we obtain the relation

$$\phi_X''(0) = -\mathbb{E}(X^2)$$
from the second formula evaluated in \( t = 0 \).

2. Skipped.

3. By the assumptions made, we obtain
\[
\mathbb{E}(e^{i\bar{t}_1(X_1+X_2)+i\bar{t}_2(X_1-X_2)}) = \mathbb{E}(e^{i\bar{t}_1(X_1+X_2)}) \mathbb{E}(e^{i\bar{t}_2(X_1-X_2)})
\]
and also
\[
\mathbb{E}(e^{i\bar{t}_1(X_1+X_2)+i\bar{t}_2(X_1-X_2)}) = \mathbb{E}(e^{i(t_1+t_2)X_1+i(t_1-t_2)X_2}) = \phi_{X_1}(t_1 + t_2) \phi_{X_2}(t_1 - t_2)
\]
so
\[
\log \phi_{X_1}(t_1 + t_2) + \log \phi_{X_2}(t_1 - t_2) = \log \mathbb{E}(e^{i\bar{t}_1(X_1+X_2)}) + \log \mathbb{E}(e^{i\bar{t}_2(X_1-X_2)}) = g_1(t_1) + g_2(t_2)
\]
proving the claim.

4. Differentiating first the equality with respect to \( t_1 \), we obtain
\[
f_1'(t_1 + t_2) + f_2'(t_1 - t_2) = g_1'(t_1)
\]
and then with respect to \( t_2 \):
\[
f_1''(t_1 + t_2) - f_2''(t_1 - t_2) = 0
\]
Setting \( t_1 = t_2 = \frac{1}{2} \) leads to \( f_1'(t) = f_2''(0) \), and setting \( t_1 = -t_2 = \frac{1}{2} \) leads to \( f_2''(t) = f_1''(0) \). As these equalities are satisfied for arbitrary \( t \in \mathbb{R} \), this says that the second derivatives of both \( f_1 \) and \( f_2 \) are constant functions, therefore that both \( f_1 \) and \( f_2 \) are polynomials of degree less than or equal to 2.

5. The assumption is that \( \log \phi_X(t) = at^2 + bt + c \) for \( t \in \mathbb{R} \). Using the hint and writing \( \mu = \mathbb{E}(X) \), \( \sigma^2 = \text{Var}(X) \), we obtain successively:
\[
e^c = \phi_X(0) = 1 \quad \text{so } c = 0
\]
\[
b = \phi_X'(0) = i\mu \quad \text{so } b = i\mu
\]
\[
2a + b^2 = \phi_X''(0) = -\mathbb{E}(X^2) = -(\mu^2 + \sigma^2) \quad \text{so } a = -\sigma^2/2
\]
Therefore, \( \phi_X(t) = e^{\mu t - \sigma^2 t^2/2} \), which is the characteristic function of a Gaussian.

6. As \( X_1, X_2 \) are independent and Gaussian, this implies that \((X_1, X_2)\) is a Gaussian vector, i.e., that \( X_1, X_2 \) are jointly Gaussian. By the assumptions made, we also have
\[
0 = \text{Cov}(X_1 + X_2, X_1 - X_2) = \text{Var}(X_1) + \text{Cov}(X_2, X_1) - \text{Cov}(X_1, X_2) - \text{Var}(X_2) = \text{Var}(X_1) - \text{Var}(X_2)
\]
so \( \text{Var}(X_1) = \text{Var}(X_2) \) [note in passing that we did not use here the assumption that \( X_1 \) and \( X_2 \) are uncorrelated]. This finally completes the proof of the result stated in part b).

**Exercise 3.** a) Using \( \psi(x) = x^2 \) or \( \psi(x) = \sigma^2 + x^2 \) in Chebyshev’s inequality leads to respectively
\[
\mathbb{P}(\{X \geq t\}) \leq \frac{\sigma^2}{t^2} \quad \text{and} \quad \mathbb{P}(\{X \geq t\}) \leq \frac{2\sigma^2}{\sigma^2 + t^2}
\]
which is not what we want. Using the hint (with \( b \geq 0 \) in order to satisfy the hypotheses), we obtain
\[
\mathbb{P}(\{X \geq t\}) \leq \frac{\mathbb{E}((X + b)^2)}{(t + b)^2} = \frac{\sigma^2 + b^2}{(t + b)^2}
\]
Optimizing over the parameter $b$, we find that best possible bound is obtained by setting $b^* = \frac{\sigma^2}{t}$ (which is non-negative), leading to

$$\mathbb{P}(\{X \geq t\}) \leq \frac{\sigma^2}{\sigma^2 + t^2}$$

b) Using Cauchy-Schwarz’s inequality with the random variables $X$ and $Y = 1_{\{X > t\}}$, we obtain

$$\mathbb{E}(X 1_{\{X > t\}})^2 \leq \mathbb{E}(X^2) \mathbb{P}(\{X > t\})$$

On the other hand, we have $\mathbb{E}(X 1_{\{X > t\}}) = \mathbb{E}(X) - \mathbb{E}(X 1_{\{X \leq t\}}) \geq \mathbb{E}(X) - t$, therefore the result.