Homework 5

Exercise 1.

a) Let X_1, X_2 be two independent Gaussian random variables such that $\mathrm{Var}(X_1) = \mathrm{Var}(X_2)$. Show, using characteristic functions or a result from the course, that $X_1 + X_2$ and $X_1 - X_2$ are also independent Gaussian random variables.

b) Let X_1, X_2 be two independent square-integrable random variables such that $X_1 + X_2, X_1 - X_2$ are also independent random variables. Show that X_1, X_2 are jointly Gaussian random variables such that $\mathrm{Var}(X_1) = \mathrm{Var}(X_2)$.

Note. Part b), also known as Darmois-Skitovic's theorem, is considerably more challenging than part a)! Here are the steps to follow in order to prove the result (but please skip the first two).

Step 1.* (needs the dominated convergence theorem, which is outside of the scope of this course) If X is a square-integrable random variable, then ϕ_X is twice continuously differentiable.

Step 2.* (quite technical) Under the assumptions made, ϕ_{X_1} and ϕ_{X_2} have no zeros (so $\log \phi_{X_1}$ and $\log \phi_{X_2}$ are also twice continuously differentiable, according to the previous step).

Step 3. Let $f_1 = \log \phi_{X_1}$ and $f_2 = \log \phi_{X_2}$. Show that there exist functions g_1, g_2 satisfying

$$f_1(t_1 + t_2) + f_2(t_1 - t_2) = g_1(t_1) + g_2(t_2) \quad \forall t_1, t_2 \in \mathbb{R}$$

Step 4. If f_1, f_2 are twice continuously differentiable and there exist functions g_1, g_2 satisfying

$$f_1(t_1 + t_2) + f_2(t_1 - t_2) = g_1(t_1) + g_2(t_2) \quad \forall t_1, t_2 \in \mathbb{R}$$

then f_1, f_2 are polynomials of degree less than or equal to 2. *Hint:* differentiate!

Step 5. If X is square-integrable and $\log \phi_X$ is a polynomial of degree less than or equal to 2, then X is a Gaussian random variable.

Hint. If X is square-integrable, then you can take for granted that $\phi_X(0) = 1$, $\phi'_X(0) = i\mathbb{E}(X)$ and $\phi''_X(0) = -\mathbb{E}(X^2)$.

Step 6. From the course, deduce that X_1, X_2 are jointly Gaussian and that $\mathrm{Var}(X_1) = \mathrm{Var}(X_2)$.

Exercise 2.*

The moment-generating function of a random variable X is defined for any $t \in \mathbb{R}$ as

$$M_X(t) = \mathbb{E}\left(e^{tX}\right).$$

(Notice that it is similar but not equal to the characteristic function of X!)

Let $X \sim \mathrm{Binomial(n, p)}$ where, recall that, the Binomial distribution with parameters (n, p) measures the probability of k successes in n independent Bernoulli trials each with parameter p.

a) Prove that for every $a \in \mathbb{R}$ and $t > 0$,

$$\mathbb{P}(X \geq a) \leq e^{-ta} M_X(t).$$

b) Show that

$$M_X(t) = (pe^t + (1-p))^n.$$

c) Using the inequality in part a) and optimizing over all $t > 0$, show that for any fixed q such that $p < q < 1$,

$$\mathbb{P}(X \geq qn) \leq \left(\frac{p}{q}\right)^{qn} \left(\frac{1-p}{1-q}\right)^{(1-q)n}.$$

d) Using the inequalities studied in the class, show that

$$\mathbb{P}(X \geq qn) \leq \frac{p}{q}$$

and compare this inequality with the one in part (c). Which inequality is better ?

Exercise 3.

a) Let X be a square-integrable random variable such that $\mathbb{E}(X) = 0$ and $\mathrm{Var}(X) = \sigma^2$. Show that

$$\mathbb{P}(\{X \geq t\}) \leq \frac{\sigma^2}{\sigma^2 + t^2} \quad \text{for } t > 0$$

Hint: You may try various versions of Chebyshev's inequality here, but not all of them work. A possibility is to use the function $\psi(x) = (x+b)^2$, where b is a free parameter to optimize (but watch out that only some values of $b \in \mathbb{R}$ lead to a function ψ that satisfies the required hypotheses).

b) Let X be a square-integrable random variable such that $\mathbb{E}(X) > 0$. Show that

$$\mathbb{P}(\{X > t\}) \geq \frac{(\mathbb{E}(X) - t)^2}{\mathbb{E}(X^2)} \quad \forall 0 \leq t \leq \mathbb{E}(X)$$

Hint: Use first Cauchy-Schwarz' inequality with the random variables X and $Y = 1_{\{X>t\}}$.