Homework 5

**Exercise 1.** a) Let $X$ be a square-integrable random variable such that $E(X) = 0$ and $\text{Var}(X) = \sigma^2$. Show that $\mathbb{P}(\{X \geq t\}) \leq \frac{\sigma^2}{\sigma^2 + t^2}$ for $t > 0$

*Hint:* You may try various versions of Chebyshev’s inequality here, but not all of them work. A possibility is to use the function $\psi(x) = (x + b)^2$, where $b$ is a free parameter to optimize (but watch out that only some values of $b \in \mathbb{R}$ lead to a function $\psi$ that satisfies the required hypotheses).

b) Let $X$ be a square-integrable random variable such that $E(X) > 0$. Show that $\mathbb{P}(\{X > t\}) \geq \frac{(E(X) - t)^2}{E(X^2)}$ for $0 \leq t \leq E(X)$

*Hint:* Use first Cauchy-Schwarz’ inequality with the random variables $X$ and $Y = 1_{\{X > t\}}$.

**Exercise 2*. Let $(X_n, n \geq 1)$ be independent random variables such that $X_n \sim \text{Bern}(1 - \frac{1}{(n+1)^\alpha})$, where $\alpha > 0$.

Let us also define $Y_n = \prod_{j=1}^{n} X_j$ for $n \geq 1$.

a) What minimal condition on the parameter $\alpha > 0$ ensures that $Y_n \xrightarrow{P} 0$ as $n \to \infty$ ?

*Hint:* Use the approximation $1 - x \simeq \exp(-x)$ for $x$ small.

b) Under the same condition as that found in a), does it also hold that $Y_n \xrightarrow{L^2} 0$ as $n \to \infty$ ?

c) Under the same condition as that found in a), does it also hold that $Y_n \xrightarrow{a.s.} 0$ as $n \to \infty$ ?

*Hint:* If $Y_n = 0$, what can you deduce on $Y_m$ for $m \geq n$ ?

**Exercise 3.** a) Show that if $(A_n, n \geq 1)$ are independent events in $\mathcal{F}$ and $\sum_{n \geq 1} \mathbb{P}(A_n) = \infty$, then $\mathbb{P}\left(\bigcup_{n \geq 1} A_n\right) = 1$

*Hints:* - Start by observing that the statement is equivalent to $\mathbb{P}\left(\bigcap_{n \geq 1} A_n^c\right) = 0$.
- Use the inequality $1 - x \leq e^{-x}$, valid for all $x \in \mathbb{R}$.

b) From the same set of assumptions, reach the following stronger conclusion with a little extra effort:

$\mathbb{P}(\{\omega \in \Omega : \omega \in A_n \text{ infinitely often}\}) = \mathbb{P}\left(\bigcap_{N \geq 1} \bigcup_{n \geq N} A_n\right) = 1$

which is actually the statement of the second Borel-Cantelli lemma.
c) Let \((X_n, n \geq 1)\) be a sequence of independent random variables such that for some \(\varepsilon > 0,\)
\[\sum_{n \geq 1} \mathbb{P}(\{|X_n| \geq \varepsilon\}) = +\infty.\]
What can you conclude on the almost sure convergence of the sequence \(X_n\) towards the limiting value 0?

d) Let \((X_n, n \geq 1)\) be a sequence of independent random variables such that \(\mathbb{P}(\{X_n = n\}) = p_n = 1 - \mathbb{P}(\{X_n = 0\})\) for \(n \geq 1\). What minimal condition on the sequence \((p_n, n \geq 1)\) ensures that
\[d1)\] \(X_n \xrightarrow{P} 0\)?
\[d2)\] \(X_n \xrightarrow{L^2} 0\)?
\[d3)\] \(X_n \xrightarrow{a.s.} 0\) almost surely?

e) Let \((Y_n, n \geq 1)\) be a sequence of independent random variables such that \(Y_n \sim \text{Cauchy}(\lambda_n)\) for \(n \geq 1\). What minimal condition on the sequence \((\lambda_n, n \geq 1)\) ensures that
\[e1)\] \(Y_n \xrightarrow{n \to \infty} 0\)?
\[e2)\] \(Y_n \xrightarrow{L^2} 0\)?
\[e3)\] \(Y_n \xrightarrow{n \to \infty} 0\) almost surely?